APPLIED ANALYSIS PRELIMINARY EXAMINATION
Aug. 23, 2001

Instructions:

You have three hours to complete this exam. Work all five problems. Please start each problem on a new page. You **MUST** prove your conclusions or show a counter-example for all problems. Write your name on your exam. Each problem is worth 20 points.

1. Let $\mathbf{F}(\mathbf{x}, \mathbf{y}) = \frac{\mathbf{x}}{\mathbf{x^2+y^2}}\mathbf{i} + \frac{\mathbf{y}}{\mathbf{x^2+y^2}}\mathbf{j}$.
   (a) Find a potential function $f(x, y)$ for $\mathbf{F}$ ($\bigtriangledown f = \mathbf{F}$) and show that $\triangle f(x, y) = 0$ except the origin.
   (b) Compute the line integral $\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \oint_C [\frac{x}{x^2 + y^2} dy - \frac{y}{x^2 + y^2} dx]$ in the counter-clockwise direction where C is the boundary curve of the parallelgram with vortices $(4, 0)$, $(0, \pi)$, $(-4, 0)$, $(0, -\pi)$.

2. Prove the existence and uniqueness of a C^1 solution to the initial value problem
$$\begin{cases} u'(t) = v^2 + t^2 u + e^t \\ v'(t) = u^2 + 3tu + e^v \\ u(0) = 1, v(0) = 0. \end{cases}$$
   for $t \in [0, \delta]$, for some $\delta > 0$.

3. Through this question $f(x)$ and $f_n(x)$ are integrable functions on $R = (-\infty, \infty)$. Provide a short proof supporting, or a counter-example negating, each of the following statements:
   (a) f_n converges to f in measure if $\lim_{n\to\infty} \int_R |f_n(x) - f(x)| dx = 0$
   (b) $\lim_{n\to\infty} \int_R |f_n(x) - f(x)| dx = 0$ if f_n converges to f in measure.
   (c) $\lim_{n\to\infty} \int_R \frac{f_n}{1 + x^2 f_n(x)} = \frac{f}{1 + x^2 f(x)} dx$ if $f_n \geq 0$ and f_n converges to f a.e. on R.

4. Let $S = \{f \in L^2(0, 2\pi) \mid ||f||_{L^2} = 1\}$ be the unit sphere in $H = L^2[0, 2\pi]$. B is the weak closure of S in H.
   (a) Show that the 0 function is in B.
   (b) Show that B is the closed unit ball in H.

5. T is a linear operator form the Hilbert space $H = L^2[0, 1]$ to itself defined by $(Tf)(x) = f(x) + \int_0^x f(t) \sin(t) dt$.
   (a) Is T a compact operator.
   (b) Find all eigenvalues of T.
   (c) Is 1 a continuous spectral point of T (hint: for $A = T - I$, either the range of A is dense in H or there is a $g(x) \neq 0$ which is orthogonal to everything in the range of A).