APPLIED ANALYSIS PRELIMINARY EXAMINATION
Aug. 23, 2002

Instructions:
You have three hours to complete this exam. Work all five problems. Please start each problem on a new page. You **MUST** prove your conclusions or show a counter-example for all problems. Write your name on your exam. Each problem is worth 20 points.

1. Define $f(x) = \Sigma_{n=1}^{\infty} n(2^{-n})x^{n-1}$ for those x for which the series converges.

   (a) Find the maximum open interval of the real line where the above series converges.

   (b) Compute $\int_{[0,1]} f(x)dx$. Justify your steps.

2. Use the contraction mapping theorem to prove the existence of a C^1 function $\phi(x, y)$ solving the equation $f(x, y, \phi(x, y)) = 10$ on $[-\delta, \delta] \times [-\delta, \delta]$ for some $\delta > 0$ with $\phi(0, 0) = 2$. Here $f(x, y, z)$ is a C^1 function on R^3 with $f(0, 0, 2) = 10$, $\frac{\partial f}{\partial x}(0, 0, 2) = -8$, $\frac{\partial f}{\partial y}(0, 0, 2) = -2$, and $\frac{\partial f}{\partial z}(0, 0, 2) = -4$.

3. Through this question $g(x)$, $f(x)$, and $f_n(x)$ are Lebesgue integrable function on $R = (-\infty, \infty)$ with dx as Lebesgue measure. Provide a short proof supporting, or a counter-example negating, each of the following statements:

   (a) f_n converges to f in measure if $\lim_{n\to\infty} \int_R |f_n(x) - f(x)|dx = 0$

   (b) $\lim_{n\to\infty} \int_R \frac{f_n(x)g(x)}{(1+f_n^2(x))}dx = 0$ if f_n converges to 0 in measure.

   (c) Prove that the function

   $$h(t) = \int_{[0,+\infty)} e^{-t^2x+t}g(x)$$

   is well defined and differentiable for $t \in (0, +\infty)$.

4. $A : H = L^2(0, 2\pi) \to H$ is defined by $(Af)(t) = \int_0^{2\pi} K(t, s)f(s)ds$ where

   $$K(t, s) = \sum_{n=1}^{23} \frac{\sin(nt)\sin(ns)}{n^2}$$

   Show that:

   (a) A is a self-adjoint and compact operator.

   (b) For what kind of function $g \in H$ is the equation $Af = g + \frac{\pi}{9}f$ solvable?

   (c) For what kind of function $g \in H$ is the equation $Af = 0.823f - g$ solvable?

5. Show that the subspace c_o is closed and nowhere dense in l^∞ ($x = (x_1, x_2, x_3, \cdots, x_i, \cdots)$ is in l^∞ if $|x_i| < M$, $i = 1, 2, 3, \ldots$ for some constant M, $x \in c_o$ if $\lim_{i\to\infty} x_i = 0$).