APPLIED ANALYSIS PRELIMINARY EXAMINATION
Jan. 15, 2004

Instructions:

You have three hours to complete this exam. Work all five problems. Please start each problem on a new page. You **MUST** prove your conclusions or show a counter-example for all problems. Write your name on your exam. Each problem is worth 20 points.

1 Let λ be a positive real number.
Define the sequence x_n, $n = 0, 1, 2, ...$, by

$$x_0 = 0$$
$$x_{n+1} = \lambda + x_n^2$$

Show that $0 < \lambda \leq \frac{1}{4}$ is a necessary condition for $\lim_{n\to\infty} x_n$ to converge. Show that this condition is also sufficient.

2 f_n and g are Lebesgue measurable functions on R, g is integrable, and f_n converges to 0 in measure. Show that

(a): $\frac{f_n(x)g(x)}{1+f_n^4(x)} \longrightarrow 0$ in m. as $n \longrightarrow \infty$

(b): $\int_R \frac{f_n(x)g(x)}{1+f_n^4(x)} d\mu_L \longrightarrow 0$ as $n \longrightarrow \infty$.

3 Let $f(x)$ be a non-negative and Lebesgue-integrable function on the real line. Prove that the function

$g(t) = \int_{(0,\infty)} e^{-tx^2} f(x)$

is well defined and continuous on $[0, \infty)$. Show that the limit $\lim_{h\longrightarrow 0^+} \frac{g(0)-g(h)}{h}$ exists and is finite if and only if $\int_{(0,\infty)} x^2 f(x)dx < \infty$

4 Prove the existence of a C^1 solution to the initial value problem $u'(t) = t^2 + u^2 + e^{4u}$, $u(0) = -1$ for $t \in [0, \delta]$, for some $\delta > 0$.

5 Show that the subset $A = \{(x_1, x_2, x_3, \cdots) \mid |x_i - 1| \leq \frac{1}{i}, i = 1, 2, 3, \cdots\} \subset l^\infty$ is totally bounded in l^∞. Is it compact?