APPLIED ANALYSIS PRELIMINARY EXAMINATION
Jan. 11, 2005

Instructions:

You have three hours to complete this exam. Work all five problems. Please start each problem on a new page. You **MUST** prove your conclusions or show a counter-example for all problems. Write your name on your exam. Each problem is worth 20 points.

1. Let $u : \mathbb{R}^2 \to \mathbb{R}$ be a C^2 function.
   (a) Show that $div\ (u\nabla u) = |\nabla u|^2 + u\Delta u$
   (b) Show that if u solves the equation
   $$\Delta u = \lambda u$$
   and vanishes at the boundary of an open disk $\mathcal{D}$, then
   $$\int\int_{\mathcal{D}} |\nabla u|^2 dxdy + \lambda \int\int_{D} u^2 dxdy = 0$$

2. Let f_n be a sequence of C^2 functions on $[0, 4]$ with $f_n(0) = f_n'(0) = 0$ for all n. If $|f_n''(x)| \leq 1$ for all n and $x \in [0, 4]$, prove that the sequence has a uniformly convergent subsequence on $(1, 2)$.

3. Let $f(x) \in L^2(R)$ and define $g(t) = \int_R f^2(x) \exp(-t|x|^2)dx$ for $t \geq 0$. Show that $g(t)$ is continuously differentiable on $[0, \infty)$ (differentiable with continuous derivative, the derivative at 0 is considered to be the derivative from right) if and only if $xf(x) \in L^2(R)$.

4. Y is a closed linear subspace of l^2, and $\xi_0 \in l^2$. Give a direct proof that there is a unique $\eta_0 \in Y$ such that $||\xi_0 - \eta_0|| \leq ||\xi_0 - \eta||$ for any $\eta \in Y$ (the following formula is very useful: $||\xi - \eta||^2 + ||\xi + \eta||^2 = 2||\xi||^2 + ||\eta||^2$).

5. The continuous map: $\mathbf{F}(x, y) = (f_1(x, y), f_2(x, y))$ from R^2 to R^2 satisfies:
   (a) $|f_i(x_2, y_2) - f_i(x_1, y_1)| < \frac{1}{2}(|x_2 - x_1| + |y_2 - y_1|)$, for $i = 1, 2$.
   (b) $\mathbf{F}$ maps the set $\Omega = (0, 1) \times (0, 1)$ into itself.

   Show that F has a unique fixed point in R^2.