Applied Analysis: Preliminary Exam
10am – 1pm, Monday Aug. 21, 2006

Problem 1: Let f be a real-valued function on the interval [0, 1] and consider four properties that f may have:

(i) f is Lipschitz continuous on [0, 1].
(ii) f is pointwise continuous on [0, 1].
(iii) f is differentiable on [0, 1].
(iv) f is uniformly continuous on [0, 1].

(a) Define each property.

(b) For each property, specify which other properties are implied by it. (No motivation is required.)

(c) Prove two of the implications listed in (b).

(d) Pick two pairs of properties and give for each pair an example of a function that satisfies one of the properties but not the other.

Problem 2: Let I denote the closed interval [0, 1], and let k be a real-valued continuous function on $I \times I$. Consider the operator K that maps a function $f \in C(I)$ to the function

$$[K\,f](x) = \int_0^1 k(x,y)\,f(y)\,dy, \qquad \text{for } x \in I.$$

(a) Prove that $K : C(I) \to C(I)$, and that K is bounded and continuous.

(b) Prove that K maps weakly convergent sequences in $C(I)$ to strongly convergent sequences in $C(I)$.

Problem 3: Let H be a Hilbert space.

(a) Give a definition of a *compact* operator on H. (Recall that there are several equivalent ways of defining compactness; please give only one.)

(b) Let $(T_n)_{n=1}^{\infty}$ be a sequence of compact operators in $\mathcal{B}(H)$ such that $||T_n|| \leq 1$ for $n = 1, 2, \ldots$. Set

$$T = \sum_{n=1}^{\infty} \frac{1}{n^2} T_n.$$

Prove directly from the definition you gave that T is a compact operator.

(c) Give an example of a Hilbert space H, and a sequence of compact operators $(S_n)_{n=1}^{\infty}$ on H such that

(i) $||S_n|| \leq 1$ for $n = 1, 2, \ldots$,
(ii) the operators $V_N = \sum_{n=1}^{N} \frac{1}{n} S_n$ converge strongly as $N \to \infty$, and
(iii) the strong limit of the operators V_N is not compact.

Problem 4: Let A be a bounded linear operator on a Hilbert space H.

(a) Define the spectrum $\sigma(A)$, and the resolvent set $\rho(A)$.

(b) Prove that if U is a unitary operator, then $\sigma(U)$ is a subset (not necessarily a proper subset) of the unit circle in the complex plane.

Problem 5: Consider the integral equation

$$u(x) - \int_0^x \big(u(t)\big)^2 \, dt = x. \tag{1}$$

(a) Prove that for some positive number α, equation (1) has a unique solution in $C([0, \alpha])$. (The number α that you give does not need to be optimal.)

(b) For the α determined in (a), how smooth is the function u on $[0, \alpha]$? Briefly motivate your answer. (A formal proof is not required.)