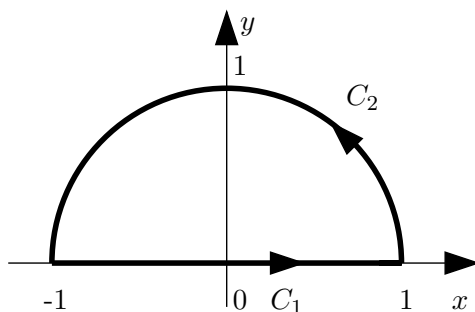


## Applied Analysis: Preliminary Exam

Tuesday, Jan. 17, 2006

**Problem 1:** In the  $xy$ -plane, let  $C_1$  denote the straight line from  $(-1, 0)$ , to  $(1, 0)$ , and let  $C_2$  denote the upper half of the unit circle, starting at  $(1, 0)$  and ending at  $(-1, 0)$ , as shown below:



Let  $\mathbf{F}$  be the vector field  $\mathbf{F}(x, y) = [2x(1 + x^2y)e^{x^2y} - y, x^4e^{x^2y} + \frac{1}{3}x^3 + xy^2]$ .

(a) Compute  $\int_{C_1} \mathbf{F}(x, y) \cdot d\mathbf{r}$  by parameterizing the integral.

(b) Use Green's theorem, and your answer from (a), to evaluate  $\int_{C_2} \mathbf{F}(x, y) \cdot d\mathbf{r}$ .

**Problem 2:** Suppose that  $(f_n)_{n=1}^{\infty}$  is a sequence of functions in  $C(\mathbb{R})$  such that  $f_n(0) = 0$ , and for each  $x \in \mathbb{R}$ ,

$$\sup_{y \neq x} \frac{|f_n(x) - f_n(y)|}{|x - y|} \leq 1 + (\sin x)^2.$$

(a) Prove that for any finite  $A$ , the sequence  $(f_n)_{n=1}^{\infty}$  has a subsequence  $(f_{n_j})_{j=1}^{\infty}$  that converges in  $C([-A, A])$ .

(b) Does the sequence  $(f_n)_{n=1}^{\infty}$  necessarily have a subsequence that converges in  $C(\mathbb{R})$ ? (Please provide a proof or a counter-example.)

**Problem 3:** Let  $(\alpha_n)_{n=1}^{\infty}$  be a sequence of real numbers that satisfy

$$\sum_{n=1}^{\infty} n^c \alpha_n^2 < \infty,$$

for some real number  $c$  such that  $c > 3$ . For  $N = 1, 2, \dots$ , set

$$f_N(x) = \sum_{n=1}^N \alpha_n \sin(nx).$$

Prove that the sequence  $(f_N)_{N=1}^{\infty}$  converges in  $C^1([- \pi, \pi])$ . (Prove the statement directly, without invoking Sobolev embedding-type theorems.)

**Problem 4:** Let  $c_0$  be the subspace of  $l^\infty$  defined as the set of all elements  $x = (x_n)_{n=1}^\infty \in l^\infty$  such that  $\lim_{n \rightarrow \infty} x_n = 0$ . Prove that the topological dual of  $c_0$  is isometrically isomorphic to  $l^1$ .

**Problem 5:** Let  $X$  denote a Hilbert space, and let  $T$  denote a self-adjoint compact operator on  $X$  such that  $\|T\| \leq 1/2$ . Define  $A_0 = I + T$ , and set, for  $n \geq 1$ ,

$$A_n = \frac{1}{2}(A_{n-1} + A_{n-1}^{-1}).$$

Prove that the sequence  $(A_n)_{n=1}^\infty$  converges in the strong sense.