Applied Analysis: Preliminary Exam
Jan. 16, 2007

Problem 1: Recall that if I is an interval in $\mathbb{R}$, then any function $f \in L^p(I)$ satisfies

$$\int_I |f(x)|^p \, dx < \infty, \qquad \text{when } p \in [1, \infty),$$
$$\operatorname*{esssup}_{x \in I} |f(x)| < \infty, \qquad \text{when } p = \infty.$$

Analogously, any sequence $x = (x_1, x_2, \dots) \in l^p$ satisfies

$$\sum_{n=1}^{\infty} |x_n|^p \, dx < \infty, \qquad \text{when } p \in [1, \infty),$$
$$\sup_{n \in \mathbb{N}} |x_n| < \infty, \qquad \text{when } p = \infty.$$

Four pairs of function spaces are given below. For each pair $[A, B]$, answer two questions:

$$A \subseteq B \ ? \qquad B \subseteq A \ ?$$

If the statement is true, then prove it. If not, then give a counter example.

(1) $[l^1, l^2]$
(2) $[l^1, l^\infty]$
(3) $[L^1(0,1), L^2(0,1)]$
(4) $[L^1(\mathbb{R}), L^2(\mathbb{R})]$

Problem 2: Let A and B be the linear operators on l^2 defined by

$$A : \ (x_1, x_2, \dots) \ \rightarrow \ (x_1, \frac{1}{2}x_2, \dots \frac{1}{2^n}x_n, \dots),$$

and

$$B : \ (x_1, x_2, \dots) \ \rightarrow \ (0, x_1, \frac{1}{2}x_2, \dots \frac{1}{n}x_n, \dots).$$

Prove that A and B are compact. Prove that B does not have any eigenvalues. Determine the spectrum of B.

Problem 3: Let S denote the collection of all simply connected subsets of $\mathbb{R}^2$ that have smooth boundaries. Define a function φ that maps a set $\Omega \in S$ to $\mathbb{R}$ via the formula

$$\varphi(\Omega) = \int_{\partial\Omega} \boldsymbol{n}(x) \cdot \boldsymbol{F}(x) \, ds(x),$$

where, for $x = (x_1, x_2) \in \mathbb{R}^2$

$$\boldsymbol{F}(x) = (4\,x - 2\,x\,y^2, \, 4\,x - x^2\,y),$$

where $\partial\Omega$ is the boundary of Ω, where for $x \in \partial\Omega$, $\boldsymbol{n}(x)$ is the outwards pointing unit normal to $\partial\Omega$ at x, and where $ds(x)$ is the element of arclength along $\partial\Omega$. Determine a subset $\tilde{\Omega} \in S$ for which

$$\varphi(\tilde{\Omega}) = \sup_{\Omega \in S} \varphi(\Omega).$$

Problem 4: Set $I = [0,1]$ and consider the Banach space $X = C(I)$ (equipped with the uniform norm, as usual). Define for $n = 1, 2, 3, \ldots$ functionals $T_n \in X^*$ via the formula

$$T_n(f) = f(1/n).$$

Prove that the set $\Omega = \{T_n\}_{n=1}^{\infty}$ is not compact in the norm topology on X^*.

Problem 5: Consider the equation

$$u(x) + u(x)^2 + \int_0^x \big(1 + \cos(x + u(y))\big)\, dy = 0. \tag{1}$$

Prove that equation (1) has a continuously differentiable solution u in some open interval around the origin.