APPLIED ANALYSIS PRELIMINARY EXAMINATION
Jan. 14, 2009

Instructions:

You have three hours to complete this exam. Work all five problems. Please start each problem on a new page. You **MUST** prove your conclusions or show a counter-example for all problems. Write your name on your exam. Each problem is worth 20 points.

1. Let D be the region given by $x \geq 0$, $y \geq 0$, $z \geq 0$, and $x + 2y + 3z \leq 6$, and let $f(x, y, z) = x^2 + 2y^2 + 3z^2$.

   (a) Show that f attains its maximum in D.

   (b) Find the maximum of f in D.

2. Evaluate the limit

$$\lim_{n \to \infty} n^2 \int_0^1 \frac{1 - \cos(x/n)}{x^2(1 + x)}\, dx.$$

   Make sure to justify your calculation.

3. Let M be a closed linear subspace of a Hilbert space H, and let $x \in X$ be an element such that

$$d = \inf_{y \in M} ||y - x|| > 0.$$

   Prove that there exists an element $y \in M$ such that $d = ||x - y||$. (This is a part of the projection theorem and you are not allowed to use it.)

4. Assume that f, $f_n \in X = C[0, 1]$ with $||f_n||_X \leq 1$, $\int_0^1 f_n(x)h(x) \longrightarrow \int_0^1 f(x)h(x)$ for any $h \in X$, and $k(x, y) \in C[0, 1] \times [0, 1]$. Let $g_n(x) = \int_0^1 k(x, y)f_n(y)dy$ and $g(x) = \int_0^1 k(x, y)f(y)dy$. Show that g_n converges to g in X.

5. Use the contraction mapping theorem to prove the existence of a C^4 function $\phi(x, y)$ solving the equation $\sin(\phi(x, y)) + (x^2 - y)\phi(x, y) + \phi^3(x, y) + x + \cos(y) - 1 = 0$ on the closed ball of radius δ and centered at $(0, 0)$ for some $\delta > 0$ with $\phi(0, 0) = 0$.