Applied Analysis: Preliminary Exam
10.00am – 1.00pm, Aug. 17, 2009

Problem 1:
Evaluate the limit

$$\lim_{n\to\infty} n^2 \int_0^1 \frac{1-\cos(x/n)}{x^2(1+x)}\,dx.$$

Make sure to justify your calculation.

Problem 2: Let f, $f_n \in X = C[0,1]$ with $||f_n||_X \leq 2$, and assume that $\int_0^1 f_n(x)h(x) \longrightarrow \int_0^1 f(x)h(x)$ for any $h \in X$. We also assume that $k(x,y) \in C[0,1]\times[0,1]$. Let $g_n(x) = \int_0^1 k(x,y)f_n(y)dy$ and $g(x) = \int_0^1 k(x,y)f(y)dy$. Show that g_n converges to g in X.

Problem 3: Let $g(x,y)$ be a continues function on R^2 satisfyiny $|g(x,y)| < 1$ for all $(x,y) \in R^2$. Show that the equation $u(x) = h(x) + \int_{(0,1)} g(x,y)sin(u(y))dy$ has a unique solution $u \in C([0,1])$ for any given $h(x) \in C^1([0,1])$.

Problem 4:
State and prove the projection lemma (Theorem 6.13 from the book).

Problem 5:
Let X denote the set of all continuously differentiable functions on the interval $I = [0, 1]$. Define for $u \in X$ the norm

$$||u|| = |u(0)| + \max_{0\leq t\leq 1} |u'(t)|.$$

(a) Prove that $||\cdot||$ is a norm on X.

(b) Prove that $\langle X, ||\cdot||\rangle$ is complete. You may use that $C(I)$ is complete with respect to the uniform norm.

(c) Prove that X is not complete with respect to the uniform norm.

(d) Let $\mathcal{P}(I)$ denote the set of all polynomial functions restricted to I. This set is dense in $\langle X, ||\cdot||\rangle$. Define for $u \in \mathcal{P}(I)$ the functionals

$$\varphi_1(u) = u(1/2), \qquad \varphi_2(u) = u'(1/2), \qquad \varphi_3(u) = u''(1/2),$$

$$\varphi_4(u) = \int_0^1 u(t)\,dt, \qquad \varphi_5(u) = u(1/2)\int_0^1 u(t)\,dt.$$

Which of these five functionals can be extended to elements in the dual space of $\langle X, ||\cdot||\rangle$?