Applied Mathematics Preliminary Examination **Aug. 24, 1993**
Numerical Analysis

Solve any **four** out of the following six problems. Each problem is 25 points if solved completely.

No books or tables can be used.

1. Let $s = \{s_n\}_{n=0}^{n=N-1}$ be a finite sequence and

$$a_k = \sum_{n=0}^{n=N-1-k} s_n\, s_{n+k}, \quad k = 0, 1, 2, \ldots, N-1,$$

its autocorrelation.

Let

$$S(z) = \sum_{n=0}^{n=N-1} s_n z^n$$

be a polynomial associated with the sequence s.

Show that

$$(*) \qquad S(z)S(1/z) = A(z),$$

where

$$A(z) = a_0 + \sum_{k=1}^{k=N-1} a_k(z^k + \frac{1}{z^k})$$

and show how to compute the sequence a_k using Discrete Fourier Transform. Hint: Consider (*) on the unit circle, $z = \exp(it)$.

2. Consider an $N \times N$ matrix of the form in which all elements are zero except in the first row and column and the main diagonal. How many operations will Gaussian elimination require if no pivoting is used? Does pivoting change the number of operations (ignore the costs associated with pivoting itself)? Find a reordering of the equations and the unknowns so that only $O(N)$ operations are required.

3. Let x be a simple zero of a twice continuously differentiable function f, $f(x) = 0$. Show that the iteration to find x ,

$$y^* = x_n - \frac{f(x_n)}{f'(x_n)}, \quad x_{n+1} = y^* - \frac{f(y^*)}{f'(x_n)},$$

is cubically convergent.

4. Show that Dufort-Frankel method

$$u_j^{m+1} - u_j^{m-1} = \frac{c\Delta t}{(\Delta x)^2}(u_{j+1}^m - u_j^{m+1} - u_j^{m-1} + u_{j-1}^m)$$

for the heat equation $u_t = cu_{xx}, \quad c \geq 0, \quad t \geq 0$ is unconditionally stable.

5. Show that implicit Euler scheme is exact for

$$y'(t) = Ay(t), \quad y(0) = y_0,$$

where

$$A = \begin{pmatrix} 0 & 3 \\ 0 & 0 \end{pmatrix}.$$

Hint: $y(t) = (I + At + A^2t^2/2 + \ldots)y_0$.

6. Show that iteration

$$y_{n+1}(t) = y_0 + \int_{t_0}^{t} f(s, y_n(s))\, ds$$

converges as $n \to \infty$ to the solution of

$$y'(t) = Ay(t), \quad y(t_0) = y_0,$$

where

$$f(s, y(s)) = Ay(s),$$

and

$$A = \begin{pmatrix} 0 & 3 \\ 0 & 0 \end{pmatrix}.$$