**Applied Mathematics Preliminary Examination** **August 23, 1994**

**Numerical Analysis**

Solve any **four** out of the following six problems.

No books or tables can be used.

1. Determine the cubic spline $S(x)$ with knots at $x = 0, 1, 2$ which interpolates the function $f(x) = x^4$. Use natural splines with the conditions $S''(0) = S''(2) = 0$.

2. Let $f(x)$ be sufficiently smooth in a neighborhood of $x = x^*$, a root of $f(x) = 0$. Show that Newton's method is quadratically convergent near $x^*$ if this root is simple. Show by example what can happen if $x^*$ is a root of multiplicity $m > 1$ and show how Newton's method can be improved for such a case.

3. Let $A$ be an $m \times n$, $m \geq n$, real rectangular matrix. Define the QR factorization and explain how the factorization proceeds using either Givens rotations or Householder reflections. Describe how it could be used to solve the general equation $Ax = b$ in the least squares sense.

4. Define the shifted inverse power method for finding the eigenvalue nearest a given value $\mu$ of a digonalizable $n \times n$ matrix $A$. Show that it converges when this eigenvalue is the unique eigenvalue closest to $\mu$.

5. Consider the equaiton

$$\frac{\partial u}{\partial t} = c\frac{\partial u}{\partial x} \quad , \quad 0 < x, t < \infty$$

$$u(0,t) = 0, \quad u(x,0) = f(x)$$

What is its characteristic direction? Describe an explicit finite difference scheme for its solution and determine its stability and convergence properties.

6. Write the ordinary differential equation

$$y''' + 2y'' + y' + 2y = 0$$

with initial conditions

$$y(0) = 1, \quad y'(0) = 0 \quad y''(0) = 0$$

as a first-order system. Find its characteristic polynomial and sketch the absolute stability region for the second order Runge-Kutta method

$$k_1 = hf(x_n, y_n)$$

$$k_2 = hf(x_n + h, y_n + k_1)$$

$$y_{n+1} = y_n + \tfrac{1}{2}(k_1 + 2k_2)$$

and the fourth order Runge-Kutta method

$$k_1 = hf(x_n, y_n)$$

$$k_2 = hf(x_n + \tfrac{1}{2}h, y_n + \tfrac{1}{2}k_1)$$

$$k_3 = hf(x_n + \tfrac{1}{2}h, y_n + \tfrac{1}{2}k_2)$$

$$k_4 = hf(x_n + h, y_n + k_3)$$

$$y_{n+1} = y_n + \tfrac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4).$$

Which should be used for this problem (explain your answer!)? How should the stepsize be chosen?