

**Program in Applied Mathematics**  
**Preliminary Exam in Numerical Analysis**

Jan 12 , 1996

Please do 5 of the following 6 questions. The test will last 3 hours.

1. **Gaussian Quadrature:** Consider the formula

$$\int_{-1}^1 f(x)w(x)dx = \sum_{i=1}^3 f(x_i)w_i + E(f),$$

where  $w(x) = |x|$ .

- (a) Find  $w_i$  and  $x_i$  for  $i = 1, 2, 3$ . (Hint: Find the  $x_i$ 's first, then find the  $w_i$ 's. Think symmetry!)
  - (b) Give an expression for  $E(f)$ , first in general terms, then specific to this application. Be as precise as possible.
2. **Singular Value Decomposition:** Let  $A$  be an  $m \times n$  real valued matrix. Let  $\underline{b} \in \Re^m$  and consider the least-squares problem: find  $\underline{x} \in \Re^n$  that minimizes

$$\|A\underline{x} - \underline{b}\|.$$

- (a) Describe the singular value decomposition (SVD) of  $A$ .
  - (b) Describe a numerical method for computing the SVD of  $A$ .
  - (c) Use the SVD to find a solution of the least-squares problem.
  - (d) Under what conditions is the solution of the least-squares problem unique? State this in terms of the SVD.
  - (e) If the solution of the least-squares problem is not unique, what properties does the particular solution obtained via the SVD possess?
3. **Multistep Methods for ODEs:** Consider the ODE initial value problem

$$y' = f(x, y)$$

and the numerical scheme

$$y_{n+1} = 4y_n - 3y_{n-1} - 2hf(x_{n-1}, y_{n-1}).$$

- (a) Determine the order of the method
- (b) Prove or disprove the stability of the method.

4. **Finite Difference Approximation to PDEs:** Let  $\alpha \geq 0$  be a constant and consider the convection-diffusion equation

$$\begin{aligned} u_t &= u_{xx} + \alpha u_x && \text{for } (x, t) \in [0, 1] \times [0, \infty), \\ u(x, 0) &= f(x) && \text{for } x \in [0, 1], \\ u(0, t) &= u(1, t) = 0 && \text{for } t > 0. \end{aligned}$$

- (a) Using backward Euler in time and central differencing in space on uniform grids, write the linear equation for the finite difference approximation  $u_k^n$  to  $u(n\delta t, k\delta x)$ .
- (b) What conditions on  $\delta t$ ,  $\delta x$ , and  $\alpha$  guarantee diagonal dominance of the associated matrix for fixed  $n$ ?
- (c) What does this condition become if it is to hold for all  $\delta t$ ?
- (d) Discuss stability of the method.
- (e) Repeat (a)-(d) when the central difference approximation of  $\alpha u_x$  is replaced by upwind differences.
5. **Variational forms:** Each of the following problems can be cast in the form  $Ax = b$ . You will be asked about the associated variational form of the problem: minimize  $P(x)$ .
- (a) Let  $A > 0$  and  $b$  be real numbers. Show that the solution to  $Ax = b$  minimizes  $P(x) = \frac{1}{2}Ax^2 - bx$ . What condition guarantees that the solution gives a minimum of  $P$ ? What is the minimum value of  $P$ ?
- (b) Let  $A$  be a  $2 \times 2$  symmetric positive definite matrix

$$A = \begin{pmatrix} a & c \\ c & d \end{pmatrix}$$

and let  $b \in \mathbf{R}^2$ . Working with components, show that the solution of  $Ax = b$  minimizes  $P(x) = \frac{1}{2}x^T Ax - x^T b$ . What is the solution? Show how the properties of  $A$  guarantee that the solution gives a minimum of  $P$ .

- (c) Let  $A$  be an  $n \times n$  symmetric positive definite matrix and let  $b \in \mathbf{R}^n$ . Arguing from the result of part(b), what functional  $P(x)$  is minimized by the solution of  $Ax = b$ ? Prove that the solution minimizes  $P(x)$ .
- (d) Let  $Ax = b$  be the two-point boundary value problem

$$\begin{aligned} -(a(t)x'(t))' &= b(t) \\ x(0) &= 0 \\ x(1) &= 0 \end{aligned}$$

where  $a(t) > 0$  on  $[0, 1]$ . Show that the solution to this boundary value problem minimizes

$$P(x) = \int_0^1 \left[ \frac{1}{2}a(t) \left( \frac{dx}{dt} \right)^2 - x(t)b(t) \right] dt.$$

6. **Secant Method:** Let  $f \in C(a, b)$  and suppose  $f(x) = 0$  has a root at  $\alpha \in (a, b)$ .
- (a) Define the secant method for finding  $\alpha$ .
  - (b) State sufficient conditions on  $f$ ,  $x_0$  and  $x_1$  for the secant method to converge to  $\alpha$ . (In essence, state and sketch a proof of a convergence theorem for the secant method.)
  - (c) What is the order of convergence of the secant method. Sketch the proof.
  - (d) Explain when the secant method is computationally superior to Newton's method.