Department of Applied Mathematics
Preliminary Examination in Numerical Analysis
Tuesday, August 18, 1998

This test contains one problem in each of six categories. Submit solutions to the problems in four (and no more) categories. The test will last from 10 am to 1 pm.

1. **Root finding:**

The almost universally used algorithm to compute $\sqrt{a}$ is the recursion $x_{n+1} = \frac{1}{2}(x_n + a/x_n)$, easily obtained by means of Newton's method.

a. Assume that we have available to us a very simple processor that only supports addition, subtraction, multiplication and halving (a subtraction of one in the exponent) but not general divide. Devise a fast algorithm for this processor to approximate $1/\sqrt{a}$ (it then only remains to multiply with a to get $\sqrt{a}$).

Also other approaches (than Newton) are available to generate fast iterations for $\sqrt{a}$. One approach starts by noting that

$$\sqrt{a} = x(1-r)^{-1/2} \qquad \text{where } r = 1 - \tfrac{x^2}{a}$$

is an identity for any (positive) value of x. Hence the iteration

$$x_{n+1} = x_n(1-r)^{-1/2} \qquad \text{where } r = 1 - \tfrac{x_n^2}{a}$$

will converge in one step to $\sqrt{a}$. However, this iteration is useless since it requires the computation of a square root - we could then as well have computed $\sqrt{a}$ directly.
Noting further that r becomes small if x_n is a good approximation to $\sqrt{a}$ suggests that we replace $(1-r)^{-1/2}$ by a truncated Taylor expansion. By including different numbers of terms, we get iterative procedures of arbitrary order of convergence.

b. Derive the quadratically convergent recursion $x_{n+1} = \dfrac{x_n}{2}(3 - x_n^2/a)$.

2. **Numerical linear algebra:**

We wish to solve the matrix equation $A\,X + X\,B = C$ where A, B, C and X are all $n \times n$ matrices; A, B, C are given, and X is unknown.

a. Illustrate the size and the sparsity structure of the linear system that one would obtain if one simply combined the separate columns of X into one single long column - and similarly for the columns of C. Roughly what would the operation count be if one applied Gaussian elimination to that system? Please answer in the form $O(n^?)$.

b. A far more effective algorithm for this matrix problem can be devised as follows: the QR algorithm can be used to quickly find Q_1 and Q_2 unitary such that $Q_1^* A\, Q_1 = A'$ is upper triangular and $Q_2^* B\, Q_2 = B'$ is lower triangular. Slight algebra now allows the matrix equation to be rewritten as

$$A'X' + X'B' = C'$$

where $X' = Q_1^* X Q_2$ and $C' = Q_1^* C Q_2$. This is a matrix problem of similar form to the original one, but with the key difference that both A' and B' now are triangular. Demonstrate that this new system can be solved by immediate back substitution, assuming no divide by zero is encountered (this can be seen to be the case if and only if $\lambda_i + \mu_j \neq 0$ for all i and j where λ_i and μ_j denote the eigenvalues of A and B, i.e. the diagonal elements of A' and B' respectively).

Hint: Start with a 3×3 system to get the idea.

3. **Discrete Fourier Transform (DFT):**

Computer programs for the complex FFT and the real FCT and FST (fast cosine- and sine-transforms respectively) usually don't bother to normalize the results, but calculate the following sums

FFT: $\hat{c}_j = \Sigma_{k=0}^{N-1} c_k e^{2is\pi kj/N}$ $\quad j = 0,1,\ldots,N-1,\quad c_k, \hat{c}_j$ complex, $s = \pm 1$ according to direction of transform

FCT: $\hat{a}_j = \Sigma_{k=0}^{N} d_k a_k \cos \frac{\pi kj}{N}$ $\quad j = 0,1,\ldots,N,\ a_k, \hat{a}_j$ real,

$$d_k = \begin{cases} 1 & \text{for } k = 0, N \\ 2 & \text{for } k = 1, 2, \ldots, N-1 \end{cases}$$

FST: $\hat{a}_j = 2\Sigma_{k=1}^{N-1} a_k \sin \frac{\pi kj}{N}$ $\quad j = 0,1,\ldots,N,\ a_k, \hat{a}_j$ real.

A very convenient feature of FCT and FST is that they are identical to their own inverses (again, ignoring normalization), i.e. there is no need to specify a transform direction. Verify analytically this feature in the case of the FST.

Hint: The following two results may be assumed known:

i. $\sin\alpha \ \sin\beta = -\frac{1}{2}\cos(\alpha+\beta) + \frac{1}{2}\cos(\alpha-\beta)$

ii. $\Sigma' \cos \frac{m\pi j}{N} = \begin{cases} N & m = 0, \pm 2N, \pm 4N, \ldots \\ 0 & \text{otherwise} \end{cases}$

where $\Sigma' = \Sigma_{j=0}^{N}$ with the first and last terms halved.

4. **Finite Elements:**

Consider Poisson's equation, $-\Delta p = f$, on the unit square with homogeneous Dirichlet boundary conditions, $p(x,y) = 0$ if $x = 0$ or $y = 0$.

a. Use the Divergence theorem to write this equation in its weak form.

b. Derive the 5-point stencil for the finite element discretization of this form on a uniform mesh based on piecewise linear elements on triangles. (You only have to compute the matrix, not the right-hand side.) Orient all of the triangles so that the hypotenuses make a 45 degree angle with the x-axis.

Hint: Consider only the two triangles that connect $x_{i,j}$ and $x_{i+1,j}$ and show that their contribution is $(p_{i+1,j} - p_{i,j}) / h$. Use symmetry for the other analogous terms.

5. **Multigrid:**

Consider the two-point boundary value problem $-u'' = f$ with periodic boundary conditions on [0,1]. Suppose the problem is discretized on a uniform grid (mesh size $h = 1/n$, mesh points $x_i = i\,h$) to form the $n{\times}n$ matrix equation $A\,u = h^2 f$, where A corresponds to the stencil [-1 2 -1] . Consider Jacobi's method

$$u_j^{(n+1)} = \tfrac{1}{2}\left[u_{j-1}^{(n)} + u_{j+1}^{(n)}\right] + \tfrac{h^2}{2} f_j$$

a. Show that the k^{th} discrete sine mode u, where $u_i = \sin(k\pi x_i)$, $k = 1,2,...,n$, is an eigenvector of both A and of the Jacibi error propagation matrix $E = I - ½\,A$.

Hint: $2\sin(a) - \sin(a+b) - \sin(a-b) = 2\,(1 - \cos(b))\sin(a)$.

b. Show that Jacobi damps all such u, except for the smoothest ($u \equiv 1$) and most oscillary ($\sin(n\pi x_i)$)- modes.

c. Show that the damped Jacobi scheme with $\omega = 2/3$ and error propagation matrix $E = I - \omega\,½\,A$ is an acceptable smoother for multigrid because it effectively damps high frequency sine modes ($k = n/2, n/2+1,...,n$).

6. **Finite Differences:**

For the heat equation $\frac{\partial u}{\partial t} = \sigma \frac{\partial^2 u}{\partial x^2}$, $\sigma > 0$, we can devise a second order (in both space and time) FD approximation with stencil shape

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Its exact form is easily found to be

$$\left[\tfrac{3}{2}\,u(x,t+k) - 2u(x,t) + \tfrac{1}{2}\,u(x,t-k)\right]/k = \sigma\,[u(x-h,t+k) - 2u(x,t+k) + u(x+h,t+k)]/h^2 \quad .$$

a. Explain out of general ODE method principles (just quoting the relevant results - no algebraic details needed) how you can immediately conclude that this scheme is unconditionally stable (as $h, k \to 0$).

b. Carry out von Neumann stability analysis to (hopefully) reach the same conclusion.