Department of Applied Mathematics
Preliminary Examination in Numerical Analysis
Friday, January 15, 1999

Submit solutions to two of the first three problems (1-3) and to two of the last three problems (4-6) (and no more). The test will last from 8 to 11 am.

1. **Root finding:**

It is well known that Newton's method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

is quadratically convergent to a simple root (under certain assumptions, including that the initial guess is close enough). In some applications, it is disadvantageous to compute the derivative. To overcome this, the secant method replaces $f'(x_n)$ in the denominator with $\frac{f(x_n)-f(x_{n-1})}{x_n - x_{n-1}}$.

a. State what the convergence rate is for the secant method (you do not need to prove it).

b. In Steffensen's method $f'(x_n)$ is instead replaced by $\frac{f(x_n+f(x_n))-f(x_n)}{f(x_n)}$. Show that this method is quadratically convergent (you may base your argument on the fact that Newton's method is known to be quadratically convergent). State whatever reasonably general assumptions you need.

c. Discuss the computational cost of each of the three methods, assuming that function and derivative evaluations are equally costly and ignoring other arithmetic costs. Rate the three methods in terms of effectiveness (assume in all cases a sufficiently close starting guess).

2. **Approximation theory:**

a. Define a cubic spline on the data (x_i, y_i), $i=1,2,...,n$.

b. Explain what is meant by a *natural cubic spline*,

c. Find the natural cubic spline that fits the data (x,y) = (-1,0), (0,1), (1,0). (Show all work).

3. Numerical quadrature:

a. Derive a two-point integration formula to approximate

$$\int_{-1}^{1} f(x)\,(1+x^2)dx$$

that is exact when $f(x)$ is a polynomial of degree 3.

b. Compute the error in this approximation when $f(x) = x^4$.

4. Numerical linear algebra:

Let M be an $n\times n$ diagonalizable matrix such that all eigenvalues of M have a non-zero real part. Consider the following iteration

$$X_{k+1} = \tfrac{1}{2}\,(X_k + X_k^{-1})$$

where

$$X_0 = M\,.$$

a. Prove that this iteration converges to the matrix S (the so-called sign function of the matrix M) such that the eigenvalues of S are ±1.

b. Show that the multiplicity of the eigenvalue +1 is equal to the number of eigenvalues of M with positive real part and the multiplicity of the eigenvalue -1 is equal to the number of the eigenvalues of M with negative real part.

Hint: The statement remains true even if the matrix is not diagonalizable, but there is a simple direct proof if this condition (M is diagonalizable) is imposed.

5. Numerical methods for ODEs:

a. Define the concept of stability domain,

b. Work out the stability domain for the backward Euler scheme,

c. The following linear multistep method

$$y_{n+1} - y_{n-1} = k\left(\tfrac{7}{3}\,f(t_n, y_n) - \tfrac{2}{3}\,f(t_{n-1}, y_{n-1}) + \tfrac{1}{3}\,f(t_{n-2}, y_{n-2})\right)$$

can be shown to be formally accurate to order 3 for solving $y' = f(t,y)$. It can also be shown that it entirely lacks a stability domain (apart from the single point at the origin). Can this scheme be used to numerically solve $y' = y$ and/or $y' = -y$? Explain.

6. **Finite differences for PDEs:**

A very simple attempt to numerically solve

$$u_t + u_{xxx} = 0 \tag{1}$$

would be to use forward Euler in time and centered, second order differences in space:

$$(u(x,t+k)-u(x,t))/k + (-\tfrac{1}{2}\, u(x-2h,t)+u(x-h,t)-u(x+h,t)+\tfrac{1}{2}\, u(x+2h,t))/h^3 = 0 \tag{2}$$

a. Explain, based on some general principle, why (2) is unconditionally unstable.

One idea to get around the instability in (2) would be to try a Lax-Friedrich - type approximation

$$(u(x,t+k) - \tfrac{1}{2}\, (u(x-h,t)+u(x+h,t)))/k + (-\tfrac{1}{2}\, u(x-2h,t)+u(x-h,t)-u(x+h,t)+\tfrac{1}{2}\, u(x+2h,t))/h^3 = 0 \tag{3}$$

b. Show that (3) is *consistent* only if $k^{-1} h^2 \rightarrow 0$ as $h, k \rightarrow 0$.

c. Show that (3) is *stable* only if $k / h^3 <$ some constant.

Hence, we conclude that also (3) is useless for solving (1) numerically.

d. Propose a numerical scheme that could be used to successfully solve (1).