Department of Applied Mathematics
Preliminary Examination in Numerical Analysis
Tuesday, August 22, 2000

Submit solutions to four (and no more) of the following six problems. The test will last from 10am to 1pm.

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1. Rootfinding

Consider the one-point iteration method $x_{n+1} = g(x_n)$ with a unique fix-point $x = \alpha \in [a, b]$.

a) Discuss sufficient conditions on $g(x)$ for convergence to the fix-point $x = \alpha$.

b) For a scheme with order of convergence $p \geq 1$ show that

$$\lim_{n\to\infty} \frac{x_{n+1} - \alpha}{(x_n - \alpha)^p} = \frac{g^{(p)}(\alpha)}{p!}.$$

c) Derive a quadratically convergent scheme to determine $a^{1/3}, a \in \mathcal{R}$.

2. Numerical Quadrature

The weights w_i for the equispaced Newton-Cotes quadrature

$$\int_a^b f(x)\,dx = \sum_{i=0}^{N} w_i f(x_i) \qquad x_i = a + ih, \quad h = \frac{b-a}{N}$$

can be deduced from the approximating polynomial for the integrand

$$P_N(x) \approx \sum_{i=0}^{N} l_i(x) f(x_i)$$

where $l_i(x)$ denotes the Lagrange polynomials.

Assuming that $a = x_0 = 0, b = x_1 = 1$ and $f(x) \in C^4[0,1]$, let $H(x)$ be the Hermite interpolation polynomial

$$H(x) = \sum_{i=0}^{1} f_i h_i(x) + \sum_{i=0}^{1} f_i' \tilde{h}_i(x)$$

where the cubic polynomials $h_i(x)$ and $\tilde{h}_i(x)$ satisfy

$$\begin{aligned} h_0(0) = 1, h_0(1) = h_0'(0) = h_0'(1) = 0, \\ h_1(1) = 1, h_1(0) = h_1'(0) = h_1'(1) = 0, \\ \tilde{h}_0(0) = \tilde{h}_0(1) = \tilde{h}_0'(1) = 0, \tilde{h}_0'(0) = 1, \\ \tilde{h}_1(0) = \tilde{h}_1(1) = \tilde{h}_1'(0) = 0, \tilde{h}_1'(1) = 1. \end{aligned}$$

a) Give explicit weights for a quadrature rule based on the above Hermite interpolation (i.e. the weights to apply to function and derivative values).

b) The result of part (a) could have been obtained immediately by taking the first few terms of a well known summation formula. Name this formula.

c) Given a uniform partition on an arbitrary interval $[a, b]$, determine the corresponding composite quadrature rule.

3. Numerical Linear Algebra

Consider the matrix

$$A = \begin{bmatrix} 3/4 & -1/4 & -1/4 \\ -1/4 & 3/4 & -1/4 \\ -1/4 & -1/4 & 3/4 \end{bmatrix}$$

and notice that it can be written as $A = I - \frac{1}{4}\underline{w}\underline{w}^t$, where $\underline{w} = (1, 1, 1)^t$.

a) Is A a Householder matrix? Explain your answer.

b) Find the spectrum of A.

c) Find A^{-1} and thus the solution of the linear system

$$A\underline{x} = \underline{b}$$

where $\underline{b} = (1, 0, 0)^t$. (Hint: It follows from the Sherman-Morrison formula that the inverse will be of the form $A = I - \beta\underline{w}\underline{w}^t$.)

d) Derive the iteration matrix, T_J, associated with the Jacobi iteration for the solution of the system in part b). What is the spectrum of T_J?

e) Derive the iteration matrix, T_G, associated with the Gauss-Seidel iteration for the same system. Name and use a well known theorem to determine a relation between the spectral radius of the T_G (which we denote as $\rho(T_G)$) and the spectral radius of T_J.

f) What is the error after two steps of the conjugate gradient method for this same system using $\underline{x}_0 = (0, 0, 0)^t$. Explain your answer. (This question does not require you to actually carry out the iterations).

4. Numerical Methods for ODE's

a) For a Linear multistep method define the concept of stability, the stability domain, and A(α)-stability.

b) Work out the stability domain for the Forward-Euler scheme.

c) Show that while the Backward-Euler scheme is A-stable, the predictor-corrector scheme involving the forward and Backward Euler is not.

5. Finite Difference Methods for PDE's

Consider the following finite difference approximations to convective-diffusion equation $u_t + au_x = bu_{xx}, b > 0$ with periodic boundary conditions:

(i) Richardson's scheme:

$$\begin{aligned}[u(x,t+k) - u(x,t-k)] + \mu[u(x+h,t) - u(x-h,t)] &= \\ 2\nu[u(x+h,t) - 2u(x,t) + u(x-h,t)]&\end{aligned}$$

(ii) Dufort Frankel's scheme:

$$\begin{aligned}[u(x,t+k) - u(x,t-k)] + \mu[u(x+h,t) - u(x-h,t)] &= \\ 2\nu[u(x-h,t) + u(x+h,t) - u(x,t+k) - u(x,t-k)]&\end{aligned}$$

with $\mu = ak/h$ and $\nu = bk/h^2$.

a) Explain using general ODE principles how you immediately conclude that Richardson's scheme is unconditionally unstable (as $h, k \to 0$).

Hint: The eigenvalues for a cyclic tridiagonal matrix with subdiagonal, diagonal and superdiagonal entries a_{-1}, a_0, a_1 is given by $\lambda_k = a_{-1}\omega^{-k} + a_0 + a_1\omega^k$ where $\omega = \exp(2\pi i/n)$, where n is the size of the matrix.

b) Show that the Dufort Frankel scheme is stable for $\mu^2 \le 1$ with no restriction on the size of ν,

Hint: The roots of the polynomial $a\lambda^2 + 2b\lambda + c = 0$ with complex coefficients a, b, c satisfy the condition $|\lambda| \le 1$ iff $|c| < |a|$ and $2|\overline{a}b - \overline{b}c| \le |a|^2 - |c|^2$.

6. Multigrid

Consider the following two-point boundary problem on the interval $[0, 1]$:

$$u'' = -f \qquad u(0) = u(1) = 0. \tag{1}$$

One possible approach is use Multigrid acceleration to solve the linear system $\mathbf{A}\mathbf{u} = h^2\mathbf{f}$ resulting from discretizing (1) on a uniform mesh (with meshpoints $x_j = jh$, and meshsize $h = 1/n$) using second order central differences. $\mathbf{A}$ then corresponds to the stencil $[-1\ 2\ -1]$. Consider using the undamped Jacobi Method

$$\mathbf{I}\ \mathbf{u}^{(n+1)} = \mathbf{M_J}\ \mathbf{u}^{(n)} + \mathbf{b}. \tag{2}$$

a) Find the explicit form of (2) and derive the expression for Jacobi error propagation.

b) Given that the kth discrete mode $u_j = \sin(k\pi x_j), k = 1, 2, \ldots, n-1$, is an eigenvector of both $\mathbf{A}$ and $\mathbf{M_J}$. Establish that the undamped Jacobi Method indeed converges.

c) Using the eigenvectors to expand the error, derive a mode amplification factor for each mode. Using this result explain why the Jacobi method is an inefficient scheme for Multi-grid acceleration.

d) Consider the damped Jacobi scheme $\mathbf{I}\mathbf{u}^{(n+1)} = \omega(\mathbf{M_j}\mathbf{u}^{(n)} + \frac{h^2}{2}\mathbf{f}) + (1-\omega)\mathbf{u}^n$, for proper omeaga $0 \le \omega \le 1$. Establish that this is an appropriate scheme for Multi-grid acceleration.