**Department of Applied Mathematics**
**Preliminary Examination in Numerical Analysis**
**Tuesday, January 16, 2001**

Submit solutions to four (and no more) of the following six problems. The test will last from 10am to 1pm.

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**1.** **Iterative Methods/Numerical Quadrature**

Consider the following procedure: Given $a_0, b_0 > 0$, then iterate

$$\begin{cases} a_{n+1} = \frac{1}{2}(a_n + b_n) \\ b_{n+1} = \sqrt{a_n b_n} \end{cases} \quad , \quad n = 0, 1, 2, \ldots$$

a. Show that the sequences $\{a_n\}, \{b_n\}$ converge from above and below respectively to the same limit (which we call $M(a_0, b_0)$).

b. Show that the convergence rate to this limit is *quadratic*.
Hint: Given what we know from part (a), it suffices for this to show that the sequence $\{a_n - b_n\}$ goes quadratically fast to zero.

It can be shown that $\frac{1}{M(1+x,1-x)} = \frac{1}{\pi}\int_0^\pi \frac{d\phi}{\sqrt{1-x^2\cos^2\phi}}$ when $-1 \le x \le 1$. Thus this *arithmetic-geometric mean* iteration (and variations of it) offers extraordinarily effective and accurate ways to evalulate a number of integrals and specials functions.

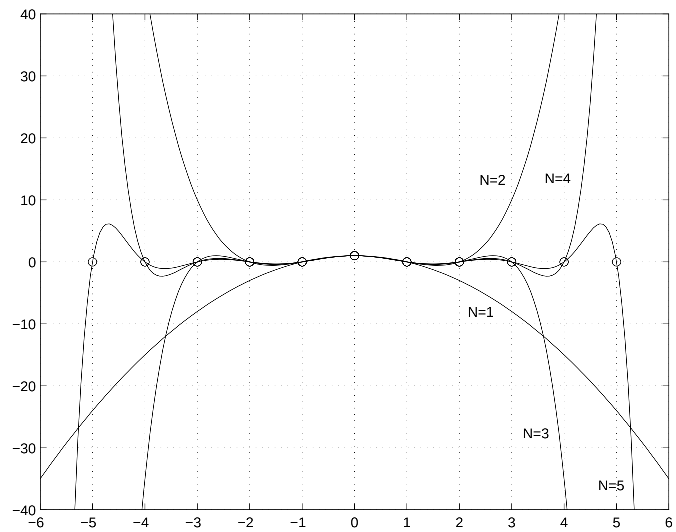
Consider next standard trapeziodal and Simpson approximations of this same integral $I = \frac{1}{\pi}\int_0^\pi \frac{d\phi}{\sqrt{1-x^2\cos^2\phi}}$ in the case of $x = 0.1$, using 3 points over the interval (i.e. with nodes located at $\phi = 0, \frac{\pi}{2}, \pi$).

c. Work out the trapezoidal approximation (by hand),

d. Work out the Simpson approximation numerically (also by hand),

e. Decide which of the two approximations is the most accurate. Explain why the answer could have been expected.

Note: For your numerical approximation, you can use $\sqrt{0.99} = 0.9950, 1/\sqrt{0.99} = 1.0050$.




**2. Interpolation**

Consider *cardinal data*

$$\begin{cases} f(0) = 1 \\ f(k) = 0 \qquad k = \pm 1, \pm 2, \ldots, \pm N \end{cases}$$

and its polynomial interpolant $p_{2N}(x)$ of degree $2N$. The figure above shows $p_{2N}(x)$ in the cases of $N = 1, 2, \ldots, 5$. Determine if, for a fixed $x$-value, $\lim_{N\to\infty} p_{2N}(x)$ exists, and if does, find its value.

Hint: The formula $\frac{\sin \pi x}{\pi x} = \Pi_{k=1}^{\infty}\left(1 - \frac{x^2}{k^2}\right)$ may come in useful.

**3. Numerical Methods for ODE's**

The second order Adams-Bashforth scheme for solving $y' = f(x, y)$ can be written

$$y_{n+1} = y_n + \frac{h}{2}[3f(x_n, y_n) - f(x_{n-1}, y_{n-1})].$$

a) Show that the boundary of the stability domain (when writing $\xi = \lambda h$) is traced by the curve

$$\xi = \frac{2r(r-1)}{3r-1} \tag{1}$$

where $r$ goes around the unit circle (i.e. $r = e^{i\theta}, -\pi \le \theta \le \pi$).

b) Demonstrate from (1) that the edge of the stability domain touches the origin in the complex $\xi$-plane, but otherwise is located entirely in the left half-plane.

**4.** **Numerical Methods for PDE's**

Consider the wave equation

$$u_{tt} = c^2 u_{xx}, \qquad x \in [0,1],\ t > 0,$$

with initial conditions $u(x,0) = f(x)$, $u_t(x,0) = g(x)$, and boundary conditions $u(0,t) = u(1,t) = 0$, for $t > 0$. Let $\Delta x = 1/N$ and define the mesh $(x_k,\ t_\ell) = (k\Delta x, \ell\Delta t)$. Let $u_k^\ell$ approximate $u(x_k,\ t_\ell)$.

a. Determine the relationship between $\Delta t$ and $\Delta x$ for which the following scheme is stable:

$$\frac{u_k^{\ell+1} - 2u_k^\ell + u_k^{\ell-1}}{\Delta t^2} = c^2 \frac{u_{k+1}^\ell - 2u_k^\ell + u_{k-1}^\ell}{\Delta x^2}.$$

Justify your answer.

**5.** **Iterative Methods**

Consider an $n \times n$ nonsingular upper triangular matrix $A$. For any right-hand-side $n$-vector $b$, we seek to solve $Ax = b$ for the unknown $n$-vector $x$. We use the iterative method

$$Mx^{(k+1)} = (M - A)x^{(k)} + b, \qquad k = 0, 1, 2, \ldots, \tag{2}$$

where $M$ is an $n \times n$ matrix to be specified.

a. Give necessary and sufficient conditions for this algorithm to converge to the solution of $Ax = b$ for any initial guess $x^{(0)}$. Justify using the convergence theory of stationary iterative methods.

b. If $M$ is diagonal, what choice of $M$ will yield the exact solution in a finite number of steps? In this case, how many steps must be taken to guarantee that $x^{(k)}$ is the exact solution (assume no round-off error).

c. Design a matrix $M$ such that the algorithm converges in one iteration. (Hint: Look at your analysis in (a) to construct $M$.)

**6.** **Linear Algebra**

a. State Gerschgorin's Theorem.

b. Define the matrix property of "diagonal dominance."

c. Prove that a diagonally dominant matrix is nonsingular.

d. Suppose A is diagonally dominant:

i) What is the equation for inverse iteration with $\lambda = 0$?

ii) Why is this equation well-defined?

iii) What assumption is needed to ensure that this iteration converges to an eigenvector of A (being as general as you can)?

iv) What is the convergence factor under this assumption?