Numerical Analysis

Answers for Preliminary Exam

August 21, 2001

1. Suppose that $\mathbf{g}(\mathbf{x})$ is a continuous vector-valued function with continuous derivatives in a neighborhood of some fixed point, α, of $\mathbf{g}(\mathbf{x})$. Assume further that the Jacobian, $\mathbf{G}(\mathbf{x})$, of $\mathbf{g}(\mathbf{x})$ satisfies $\|\mathbf{G}(\alpha)\|_\infty < 1$.

   What can you conclude about the fixed–point problem $\mathbf{x} = \mathbf{g}(\mathbf{x})$ and the fixed–point iteration $\mathbf{x}_{n+1} = \mathbf{g}(\mathbf{x}_n)$ in a neighborhood the fixed point α?

   Newton's method for solving $\mathbf{f}(\mathbf{x}) = \mathbf{0}$ can be considered as a fixed-point problem by setting $\mathbf{g}(\mathbf{x}) = \mathbf{x} - \mathbf{F}(\mathbf{x})^{-1}\mathbf{f}(\mathbf{x})$. Apply your conclusions to show that the convergence rate of Newton's method is at least quadratic.

   Answer:

   There exists a closed, bounded, convex region, D, containing α in which the fixed–point problem has a unique solution (namely, α) and in which the fixed–point iteration converges as follows whenever $\mathbf{x}_0 \in D$:

   $$\|\alpha - \mathbf{x}_{n+1}\|_\infty \le (\|\mathbf{G}(\alpha)\|_\infty + \epsilon_n)\|\alpha - \mathbf{x}_n\|_\infty,$$

   where $\epsilon_n \longrightarrow 0$ as $n \longrightarrow \infty$.

   This can be applied to Newton's method by setting $\mathbf{g}(\mathbf{x}) = \mathbf{x} - \mathbf{F}(\mathbf{x})^{-1}\mathbf{f}(\mathbf{x})$ and noting that $\mathbf{G}(\alpha) = I - \mathbf{F}(\alpha)^{-1}\mathbf{F}(\alpha) + \mathbf{0} = \mathbf{0}$. The conclusion is that Newton's method converges *superlinearly* to a zero, α, of $\mathbf{f}(\mathbf{x})$ whenever $\mathbf{F}(\alpha)$ is nonsingular and $\mathbf{f}(\mathbf{x})$ and its first and second partial derivatives are continuous in a ball about α.

2. Derive Simpson's rule and its error formula for approximating $\int_a^b f(x)dx$ using the values of f at a, $(a+b)/2$, and b.

   Then derive the composite Simpson rule and its error formula for approximating $\int_a^b f(x)dx$ using values of f at $a + jh, j = 0, 1, ..., n$, where $h = (b-a)/n$.

   Answer: See pages 256–258 of Atkinson. Key points:

   Error of the Simpson rule

   $$E_2(f) = -\frac{h^5}{90} f^{(4)}(\eta), \quad \eta \in [a, b]$$

   Error of the composite Simpson rule

   $$E_n(f) = -\frac{h^4(b-a)}{180} f^{(4)}(\eta), \quad \eta \in [a, b]$$

3. Let $A = \{a_{ij}\}_{i,j=1,\ldots,n}$ be a square complex-valued matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$. Show that if A is normal, i.e., $AA^* = A^*A$, where A^* denotes the adjoint matrix, then

   a) A is diagonalizable by a unitary matrix
   b) $\sum_{i,j=1}^{n} |a_{ij}|^2 = \sum_{j=1}^{n} |\lambda_j|^2$
   c) There is an orthonormal set of n eigenvectors of A.

   Answer: Using Schur's decomposition, there exists a unitary matrix U such that $T = UAU^*$ is upper triangular. Since A and A^* commute, so do T and T^*. This implies that T is actually a diagonal matrix.

   The Frobenius norm is invariant under a unitary transformation (either statement of the theorem or the proof).

   Columns of U are eigenvectors and they are orthonormal.

4. Obtain the minimax first degree polynomial for $f(x) = 1/(1 + x)$ on $[0, 1]$.

   Formulate the theorem describing properties of the minimax error.

   Answer: Since $f(x)$ is monotone, the answer is obtained by directly matching the requirements of Chebyshev Equioscillation theorem, Atkinson page. 224, resulting in

   $q(x) = -\frac{x}{2} + \frac{1}{\sqrt{2}} + \frac{1}{4}$

5. Describe the steps of the shifted inverse power method for finding the eigenvalue λ nearest a given value μ of a diagonalizable $n \times n$ matrix A.

   Show that the method converges when λ is the unique eigenvalue closest to μ.

   Answer: Consider $(A - \mu I)^{-1}$ and apply the power method to this matrix.

   Need to solve at each step.

6. The general form of an explicit Runge-Kutta (RK) method for solving the initial value problem

$$y' = f(t, y), \quad y(t_0) = y_0\,,$$

   over the time interval $t_{n+1} = t_n + h$ is

$$y_{n+1} = y_n + \sum_{i=1}^{p} \gamma_i d^{(i)}$$

with

$$\begin{aligned} d^{(1)} &= hf(y_n, t_n) \\ d^{(2)} &= hf(t_n + \alpha_2 h, y_n + a_1 d^{(1)}) \\ d^{(3)} &= hf(t_n + \alpha_3 h, y_n + a_2 d^{(1)} + a_3 d^{(2)}) \\ &\vdots \end{aligned}$$

and p being the order of the method.

Consider the ODE problem

$$\frac{d}{dt} u_k(t) = iku_k(t)$$

resulting from solving the 1D periodic wave propagation problem $(\partial_t + \partial_x)u = 0$ on the interval $[0, 2\pi)$ using the spectral representation $u(x,t) = \sum_k u_k(t)\exp(ikx)$.

a) Determine which of the RK methods for order $p \leq 3$ can be used to solve this problem in a stable manner. Explain your answer by showing your calculation.

b) Given that one of the conditions needed to achieve order $p = 3$ is $\alpha_2^2\gamma_2 + \alpha_3^2\gamma_3 = 1/3$, determine the remaining three conditions.

Hint: Use the fact that the stability boundary in the complex ξ- plane for explicit RK methods with order of accuracy $p \leq 3$ is determined by the polynomial

$$r(\xi) = 1 + \frac{\xi}{1!} + \cdots + \frac{\xi^p}{p!}.$$

Answer: a) To solve the ODE problem

$$\frac{\partial u}{\partial t} = iku$$

in a numerical stable manner using the explicit RK method for $p \leq 3$, the eigenvalue $\xi^* = ik\Delta t := i\eta$ must lie within or on the boundary of the stability domain in complex xi–plane. Since $\xi^* = ik\Delta t$ is pure imaginary, one must ensure that the stability domain contains a finite interval the imaginary axis. This constitutes a necessary condition for stability.

The boundary of the stability domain is given by

$$|r| = 1 \tag{0.1}$$

It follows then that

RK1; $r = 1 + i\eta$ and (1) implies $1 = 1 + \eta^2 \quad \rightarrow \quad \eta \equiv 0.$

RK2; $r = (1 - \eta^2/2) + i\eta$ and (1) implies $1 = (1 - \eta^2/2)^2 + \eta^2 \quad \rightarrow \quad \eta \equiv 0$.

RK3; $r = (1-\eta^2/2)+i\eta(1-\eta^2/6)$ and (1) implies $1 = (1-\eta^2/2)^2+\eta^2(1-\eta^2/6)^2 \quad \rightarrow$ $\eta = 0$ or $\eta = \pm\sqrt{3}$.

This only RK3 contains the imaginary axis with interval $[-\sqrt{3}, \sqrt{3}]$. Therefore this is ths only stable scheme for $p \leq 3$.

b) To find the remaining order conditions solve IVP problem $y' = \lambda y$, $y(t_0) = y_0$ with difference solution $y_n = r^n$. Thus

$d^{(1)} = hf(y_n) = \xi y_n$
$d^{(2)} = hf(y_n + a_1 d^{(1)}) = \xi(1 + a_1\xi)y_n$
$d^{(3)} = hf(y_n + a_2 d^{(1)} + a_3 d^{(2)}) = (1 + a_2\xi + a_3\xi(1 + a_1\xi))y_n$
with
$y_{n+1} = [1 + \gamma_1\xi + \gamma_2\xi(1 + a_1\xi) + \gamma_3(1 + a_2\xi + a_3\xi(1 + a_1\xi))]y_n$
$= [1 + (\gamma_1 + \gamma_2 + \gamma_3)\xi + (a_1\gamma_2 + (a_2 + a_3)\gamma_3)\xi^2 + (a_1 a_2 \gamma_3 \xi^3]y_n$

Thus, the order conditions follow on setting $y_n = r^n$ and comparing coefficients of ξ with the given polynomial, that is
$r = [1 + (\gamma_1 + \gamma_2 + \gamma_3)\xi + (a_1\gamma_2 + (a_2 + a_3)\gamma_3)\xi^2 + (a_1 a_2 \gamma_3)\xi^3]$
$\equiv 1 + \xi + \xi^2/2 + \xi^3/6$.