# Department of Applied Mathematics
# Preliminary Examination in Numerical Analysis
# Monday, August 18, 2003

Submit solutions to four (and no more) of the following six problems.

**Root finding:**

1. Consider the fixed point iteration method

$$x_{n+1} = g(x_n). \tag{1}$$

a. State the necessary conditions for existence and uniqueness of a fixed-point $x = \alpha$ in (1), and deduce the criteria that determines the order of convergence.

b. Consider instead the fixed-point iteration

$$x_{n+1} = G(x_n) = x_n - \frac{(g(x_n) - x_n)^2}{g(g(x_n)) - 2g(x_n) + x_n} \quad .$$

Show that if $\alpha$ is a fixed-point of $g(x)$, then it also a fixed point of $G(x)$.

c. Consider the function $g(x) = x^2$, and deduce the convergence properties for both fixed point methods around the roots $x = 0$ and $x = 1$.

**Numerical Quadrature:**

2. Show that the quadrature formula $\int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}} dx = \frac{\pi}{n} \sum_{k=0}^{n-1} f(\cos \pi \frac{2k+1}{2n})$ is exact for all ploynomials of degree up to and including $2n$-1.

**Interpolation / Approximation:**

3. The most common type of spline is a cubic spline. We consider here instead quadratic splines. Let a quadratic $B$-spline has its leftmost node at $x = 0$, and subsequent nodes at consecutive integer locations.

a. Determine its width of the quadratic $B$-spline described above. Assuming we normalize it so that its area is one, what are the values at the node points?

b. Determine a quadratic spline that satisfies the infinite data set $y(k) = \begin{cases} 1 & k = 0 \\ 0 & k = \pm 1, \pm 2, \pm 3, \ldots \end{cases}$. Explain if this spline is uniquely determined or not.

**Linear Algebra:**

4. a. State and apply the Gerschgorin Theorem to the matrix

$$A = \begin{pmatrix} 1 & \varepsilon & \varepsilon \\ \varepsilon & 2 & \varepsilon \\ \varepsilon & \varepsilon & 2 \end{pmatrix} \quad \text{with } \varepsilon << 1.$$

It is often possible to improve the bounds on an eigenvalue estimate by first applying a similarity transformation of $A$ involving the diagonal matrix $D_n = d_1,\ldots,d_n$.

b. Establish that a reduction in the error bound of one eigenvalue must occur at the expense of relaxing the bounds of the remaining eigenvalues.

c. Let $D_3 = (1, k\varepsilon, k\varepsilon)$, show that the bounding radius for $\lambda_1 \approx 1$ can be reduced from $\rho_1 = 2\varepsilon$ to $2\varepsilon^2$.

**Numerical ODE:**

5. Consider the linear multistep method $y_{n+1} = \frac{1}{2}(y_n + y_{n-1}) + \frac{k}{4}(4f_{n+1} - f_n + 3f_{n-1})$ for solving the ODE $y' = f(x, y)$.

a. Determine the accuracy of the method.

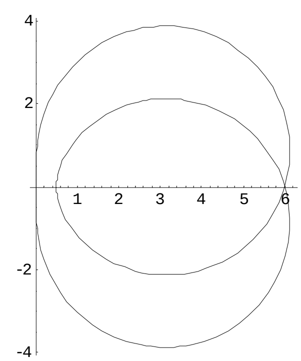
b. Determine if it satisfies the root condition (for convergence in the limit of $k \to 0$).

c. Determine the method's stability domain. What parts of the complex plane are inside it and what parts are outside it? Does it appear that the method is $A$-stable?

Hint: You may refer to the diagram to the right, which displays the complex trajectory of

$$f(\theta) = \frac{e^{2i\theta} - \frac{1}{2}e^{i\theta} - \frac{1}{2}}{e^{2i\theta} - \frac{1}{4}e^{i\theta} + \frac{3}{4}} = i\theta + \frac{5i\theta^3}{12} + \frac{\theta^4}{18} + O(\theta^5)$$

for $-\pi \le \theta \le \pi$:



## Numerical PDE:

6. Consider the initial value problem for the heat equation

$$u_t = u_{xx}.$$

Assuming periodic boundary conditions, discretize this equation in space (only) using the second order finite difference scheme. Let's denote by $A$ the matrix of the resulting system of linear ODEs. For the system of ODEs of the form $\boldsymbol{y}' = \boldsymbol{f}(t,\boldsymbol{y})$, consider the explicit midpoint method in time,

$$\mathbf{y}_{n+1} = \mathbf{y}_{n-1} + 2h\,\mathbf{f}(t_n, \mathbf{y}_n),$$

and apply it to the heat equation.

Find the stability region of the explicit midpoint method, and use it to explain the behavior of the scheme.

- Which property should matrix $A$ have in order to make the explicit midpoint method useful?
- Which type of linear PDEs gives rise to matrices with this property?