Department of Applied Mathematics
Preliminary Examination in Numerical Analysis
Friday, January 16, 2004

Submit solutions to four (and no more) of the following six problems.

Root finding:

1. Consider applying Newton's method to a real cubic polynomial.

   a. In case the polynomial has three distinct real roots $x = \alpha$, $x = \beta$, and $x = \gamma$, show that the starting guess $x_0 = \frac{1}{2}(\alpha + \beta)$ will yield γ in one step.

   b. Give a heuristic (e.g. geometric) argument showing that if two roots coincide (say, $\beta = \gamma$), there is precisely one starting guess x_0 (other than the double root) for which Newton will fail, and that this one separates the basins of attraction for the distinct roots.

   c. Extend the argument in part b to the case when all the three roots again are distinct. Explain why there now are infinitely many starting guesses x_0 for which the iterations will fail.

Numerical Quadrature:

2. Determine the nodes and weights for the 3-point Gaussian quadrature formula for the integral

$$\int_{-1}^{1} \sqrt{1-x^2}\, f(x)\, dx$$

Hint: The orthogonal polynomials $U_n(x)$ with respect to weight function $\sqrt{1-x^2}$ can be generated with the three-term recursion relation $U_{n+1}(x) = 2x\, U_n(x) - U_{n-1}(x)$, $n = 1, 2, \ldots$, together with the initial values $U_0(x) = 1$, $U_1(x) = 2x$.

Interpolation / Approximation:

3. Consider using linear interpolation to approximate $f(x) = e^{-x}$ on [0,1] with accuracy 10^{-8} using a table of function values at equally spaced points.

   a. Find the largest spacing for which it is possible.

   b. If we instead use cubic interpolation, find the corresponding largest spacing. What is an approximate factor by which the spacing can be reduced by this change from linear to cubic interpolation?

Linear Algebra:

4. Let A be an real $m \times n$ matrix and b be in R^m .

   a. Discuss the three possible cases for solvability of $Ax = b$.

   b. Show that the least squares problem of minimizing $\|Ax - b\|_2$ over x in R^n is equivalent to solving the normal equations, $A^*Ax = A^*b$.

   c. There are only two cases for solvability of the normal equations. What two and why?

Numerical ODE:

5. a. Define the truncation error for the following two-step method for solving $y' = f(x, y)$

$$y_{n+1} = a_0 y_n + a_1 y_{n-1} + h\,(b_{-1}\, f(x_{n+1}, y_{n+1}) + b_0\, f(x_n, y_n) + b_1\, f(x_{n-1}, y_{n-1})).$$

   b. Find conditions on the coefficients a_0, a_1, b_{-1}, b_0, b_1 to make it a third order method.

   c. What additional restriction is needed to make such a method stable?

Numerical PDE:

6. a. State the Lax Equivalence Theorem.

   b. Consider the following finite difference approximation

$$\frac{u(x,t+k)-u(x,t)}{k} = \frac{u(x-h,t)-u(x,t+k)-u(x,t)+u(x+h,t)}{h^2}\ .$$

   Use the Lax Equivalence Theorem to determine under what condition on k and h the solutions to this scheme will this converge to a solution to the heat equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$?