Department of Applied Mathematics
Preliminary Examination in Numerical Analysis
Tuesday August 17, 2004

Submit solutions to four (and no more) of the following six problems.

<u>Root finding:</u>

1. The sign function is defined as

$$sign(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases} .$$

It can be evaluated via an iteration which is useful for some problems. One such iteration is given by

$$x_{k+1} = \frac{3x_k - x_k^3}{2}. \qquad (1)$$

a. Find all fix points of this iteration. How many fix points are stable?

b. Find the order of convergence of the iteration at the stable fix point(s).

c. Show that for $x \in [-1, 1],\ x \neq 0,$ the iteration (1) always converges.

<u>Numerical quadrature:</u>

2. Gaussian quadratures

$$\int_{-1}^{1} f(x)\,dx = \sum_{k=0}^{N} w_k\, f(x_k)$$

with nodes including the end points (x_0 = -1 and x_N = 1) are called Gauss-Legendre-Lobatto quadratures.

a. Show that if the interior nodes $x_1, \dots, x_{N-1}$ in the quadrature are given by the roots of $p'_N(x)$, where $p_N(x)$ denotes the N:th degree Legendre polynomial, then the quadrature is exact for polynomials up to degree 2N-1.

Hint: The 2N-1:th degree Legendre polynomial can be written as

$$p_{2N-1}(x) = s_{N-2}(x)(x^2 - 1)p'_N(x) + r_N(x)$$

where $s_{N-2}(x)$ and $r_N(x)$ are N-2:th and N:th degree polynomials, respectively.

b. Find the 4-point (N=3) Gauss-Legendre_Lobatto quadrature (nodes and weights) for the integrals $\int_{-1}^{1} f(x)\,dx$.

Hint: The three term recursion for Legendre polynomials is given by

$$P_0(x) = 1,\ P_1(x) = x \ \text{ and } \ (k+1)P_{k+1} - (2k+1)\,x\,P_k + k\,P_{k-1} = 0.$$

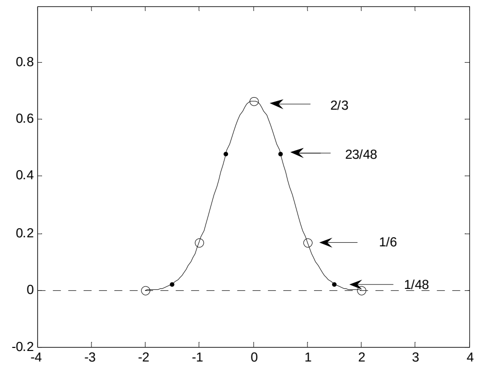
Interpolation / Approximation:

3. a. Define what is meant by a cubic spline.

b. Show that giving just function values $f(x_i)$, $i = 1, \ldots, N$ is insufficient to determine uniquely a cubic spline through the points. Work out how many additional conditions that are required for this.

c. Two ways to obtain a unique cubic spline interpolant are known as 'natural spline' and 'not-a-knot' spline. Define these two spline types.

d. A cubic B-spline is the cubic spline that is non-zero over the narrowest interval. In the case of a unit-spaced grid, it will look as shown in the figure to the right; non-zero over four subintervals only (taking values at the nodes and at subinterval midpoints as shown, when normalized in the usual fashion).



If one knows the values of such a B-spline $f(x)$ only at the node points, one can very easily tabulate it at increasingly denser sets of point locations by means of repeatedly using the relation

$$f(x) = \tfrac{1}{8}(f(2x-2) + 4f(2x-1) + 6f(2x) + 4f(2x+1) + f(2x+2)) \qquad (2)$$

Show that (2) holds true for all values of x.

Hint: One way to solve the problem, requiring only little algebra, is to let $f(x)$ in the right hand side of (2) be the cubic B-spline, and then show that uniqueness of cubic splines implies that the whole expression must evaluate to the very same spline.

Linear algebra:

4. Let A be an $m \times n$ matrix, with $m \geq n$, and $\mathbf{b}$ an $m \times 1$ vector. Consider the least-squares problem

$$\min_{\mathbf{x} \in \Re^n} \|A\mathbf{x} - \mathbf{b}\|^2,$$

which can be written as

$$A\mathbf{x} \simeq \mathbf{b}. \qquad (3)$$

a. Explain why any solution of the normal equations is a solution of this problem, regardless of the rank of A. Show that the solution is unique if the rank of A is n.

b. Describe the solution of this problem in terms of the singular value decomposition of A.

Consider the regularized least-squares problem

$$\min_{\mathbf{x} \in \Re^n} (\|A\mathbf{x} - \mathbf{b}\|^2 + \lambda^2 \|\mathbf{x}\|^2).$$

c. Derive a form of the normal equations for this problem. Use it to argue that the regularized problem has a unique solution whenever $\lambda \neq 0$, regardless of the rank of A.

Hint: Write the regularized least-square problem in the form of equation (3) using an $(m+n) \times n$ linear system.

d. Write the solution of the regularized problem in terms of the singular value decomposition of A.

e. Assume that A has only one zero singular value and $\mathbf{b}$ is not orthogonal to the corresponding left singular vector of A. Describe the effect on the solution, $\mathbf{x}$, of the regularized problem as $\lambda \to 0$.

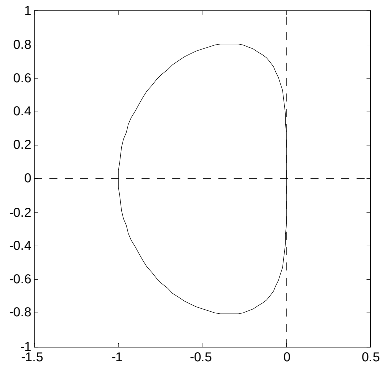
Numerical ODE:

5. A general linear multistep formula for solving $y' = f(t, y)$ takes the form

$$y_{n+1} = \sum_{j=0}^{p} a_j y_{n-j} + k \sum_{j=-1}^{q} b_j f(t_{n-j}, y_{n-j}).$$

a. Derive the coefficients a_j and b_j for the second order Adams-Bashforth and Adams-Moulton methods.

b. The figure to the right shows the stability domain for the second order Adams-Bashforth method (interior of the closed curve). Can this method be used to solve $y' = y$, $y' = -y$, and/or $y' = iy$? Explain!



c. Determine the stability domain for the second order Adams-Moulton method.

Numerical PDE:

6. Consider Richardson's difference scheme for the heat equation $u_t = u_{xx}$:

$$\frac{1}{2k}\big(u(x, t+k) - u(x, t-k)\big) = \frac{1}{h^2}\big(u(x-h, t) - 2u(x, t) + u(x+h, t)\big) .$$

a. Show that this scheme has second-order truncation error.

b. Use either ODE principles or von Neumann analysis to show that this scheme is unconditionally unstable.

c. Demonstrate a minor modification of the left-side of Richardson's scheme that yields a familiar unconditionally stable scheme and prove it.