

**Department of Applied Mathematics**  
**Preliminary Examination in Numerical Analysis**  
**Wednesday January 12, 2005 (10 am - 1 pm)**

Submit solutions to four (and no more) of the following six problems. Justify all your answers.

**Root finding:**

1.
  - a. Derive the secant method for solving one scalar equation  $f(x) = 0$ .
  - b. Show that the error in the secant method decays like  $\varepsilon_{n+1} = O(\varepsilon_n^a)$ , where  $a = \frac{\sqrt{5}+1}{2} \approx 1.6180$ .
  - c. The following is simple generalization of the secant method to a system of two equations in two unknowns
 
$$\begin{cases} f(x,y) = 0 \\ g(x,y) = 0 \end{cases} : \text{ Given three past iterates with corresponding } f \text{ and } g \text{ values, i.e. } (x_k, y_k, f_k, g_k),$$
 $k = n, n-1, n-2$ , one can fit a plane through the  $(x_k, y_k, f_k)$ - points and another one through the  $(x_k, y_k, g_k)$ -points. Each of these planes intersect the  $(x,y)$ -plane along a straight line. The next iterate will be where these two straight lines cross.

The formula for the 1-D secant method can be written as  $x_{n+1} = \left| \begin{array}{cc} x_n & f_n \\ x_{n-1} & f_{n-1} \end{array} \right| / \left| \begin{array}{c} 1 & f_n \\ 1 & f_{n-1} \end{array} \right|$ . Sketch a proof that, under the proper hypothesis, the 2-D version becomes

$$x_{n+1} = \left| \begin{array}{ccc} x_n & f_n & g_n \\ x_{n-1} & f_{n-1} & g_{n-1} \\ x_{n-2} & f_{n-2} & g_{n-2} \end{array} \right| / \left| \begin{array}{cc} 1 & f_n & g_n \\ 1 & f_{n-1} & g_{n-1} \\ 1 & f_{n-2} & g_{n-2} \end{array} \right|, y_{n+1} = \left| \begin{array}{ccc} y_n & f_n & g_n \\ y_{n-1} & f_{n-1} & g_{n-1} \\ y_{n-2} & f_{n-2} & g_{n-2} \end{array} \right| / \left| \begin{array}{cc} 1 & f_n & g_n \\ 1 & f_{n-1} & g_{n-1} \\ 1 & f_{n-2} & g_{n-2} \end{array} \right|. \quad (1)$$

Hint: The equations (1) can be written as  $\left| \begin{array}{ccc} x_n - x_{n+1} & f_n & g_n \\ x_{n-1} - x_{n+1} & f_{n-1} & g_{n-1} \\ x_{n-2} - x_{n+1} & f_{n-2} & g_{n-2} \end{array} \right| = 0, \left| \begin{array}{ccc} y_n - y_{n+1} & f_n & g_n \\ y_{n-1} - y_{n+1} & f_{n-1} & g_{n-1} \\ y_{n-2} - y_{n+1} & f_{n-2} & g_{n-2} \end{array} \right| = 0$ .

One way to arrive at these relations is to let the two planes be  $ax + \beta y + z = \gamma$  and  $\delta x + \varepsilon y + z = \zeta$ , and then substitute into these equations the points that are located on each of the planes. The result should then follow quickly.

**Numerical quadrature:**

2. Consider the singular integral  $I := \int_0^1 r^{-\alpha} dr$  for  $0 < \alpha < 1$ .
  - a. Since the integrand is concave, the right-hand rule satisfies the following bound

$$I_R := \sum_{j=1}^N h(jh)^{-\alpha} \leq 1$$

where  $h = 1/N$ . This same rule can be viewed as a left-hand rule for the integral on the interval  $(h, 1+h)$  which yields the bound

$$\int_h^{1+h} r^{-\alpha} dr \leq I_R.$$

Use this information to develop a bound on the error  $E_R = |I - I_R|$ . Find a simple lower bound on the error.

- b. Use the same technique to develop an upper bound on the error for the midpoint rule.

$$I_M := \sum_{j=1}^N h(j - \frac{1}{2})h)^{-a}$$

- c. Let  $f(r) \in C^4$ . Find a transformation on the problem that allows the use of standard quadrature rules for the integral

$$\int_0^1 r^{-a} f(r) dr.$$

Write the error bounds for the composite trapezoidal rule on the transformed integral. What transformation assures  $O(h^2)$  convergence?

**Interpolation / Approximation:**

3. a. Consider a function  $f(x)$  sampled at  $x = 0, 1, 2, 3$  with the values  $f(0) = 0, f(1) = 1, f(2) = 0$  and  $f(3) = -1$ . Find the 3rd degree polynomial that fits the data.

Using  $B$ -splines, we can piecewise interpolate the data  $s_0, \dots, s_N$  given at  $x = 0, \dots, N$  by

$$s(x) = \sum_{k=0}^N B^{(m)}(x-k) s_k \tag{2}$$

with the  $B$ -splines of order  $m+1$  given by

$$B^{(m)}(x) = \sum_{l=0}^{m+1} (-1)^l \binom{m+1}{l} \frac{(x + (m+1)/2 - l)_+^m}{m!} \tag{3}$$

and  $x_+^m$  is defined as  $x^m$  for  $x > 0$  and zero for  $x \leq 0$ .

- b. Using the properties of  $B$ -splines, relate the smoothness of  $s(x)$  in (2) to the integer  $m$  in (3). In other words, for a given  $m$ , what is the differentiability of  $s(x)$ ?
- c. Write down the expression and draw a picture of the  $B$ -spline used for piecewise linear interpolation of a data set.
- d. Compute the Fourier transform of a function obtained by piecewise linear spline interpolation of a data set, and show that it decays as  $O(\omega^2)$ .

Hint: You can use that if  $g(x)$  is real and even, then its Fourier transform is  $\hat{g}(\omega) = \int_{-\infty}^{\infty} g(x) e^{2\pi i \omega x} dx = 2 \operatorname{Re} \left( \int_0^{\infty} g(x) e^{2\pi i \omega x} dx \right)$ .

**Linear algebra:**

4. The speed by which one can solve a linear system  $Ax = b$  can depend on both the sparsity structure and the actual values in the coefficient matrix. Describe the fastest non-iterative such methods you can devise for the matrices below, and tell what the operation count will be (give your operation count answers in forms like  $O(N^2)$ ,  $O(N \log N)$ , etc., when the matrix size is  $N \times N$ ). You can assume that no singularities or other breakdowns will occur (i.e. pivoting can be used freely for the sake of speed, but you need not consider pivoting for stability or accuracy.)

a. 
$$A = \begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_{N-1} \\ a_{N-1} & a_0 & a_1 & a_2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_1 & \cdots & a_{N-2} & a_{N-1} & a_0 \end{bmatrix} \quad (\text{circulant matrix})$$



**Numerical PDE:**

6. Consider the wave equation

$$u_{tt} = c u_{xx}, \quad x \in [0, 1], \quad t > 0$$

with initial conditions  $u(x,0) = f(x)$ ,  $u_t(x,0) = g(x)$ , and boundary conditions  $u(0,t) = u(1,t) = 0$ , for  $t > 0$ . Let  $\Delta x = 1/N$  and define the mesh  $(x_k, t_l) = (k\Delta x, l\Delta t)$ . Let  $u_k^l$  approximate  $u(x_k, t_l)$ . Use von Neumann analysis to determine the relationship between  $\Delta t$  and  $\Delta x$  for which the following scheme is stable:

$$\frac{u_k^{l+1} - 2u_k^l + u_k^{l-1}}{\Delta t^2} = c \frac{u_{k+1}^l - 2u_k^l + u_{k-1}^l}{\Delta x^2}.$$