Department of Applied Mathematics
Preliminary Examination in Numerical Analysis
Thursday January 19, 2006 (10 am - 1 pm)

Submit solutions to four (and no more) of the following six problems. Justify all your answers.

Root finding:

1. a. Let A in $R^{n\times n}$ be symmetric positive definite and consider the quadratic functional $\psi(x)=\frac{1}{2}<x,Ax>-<x,b>$. What is the iteration formula for steepest descent applied to $Ax=b$?

b. Use calculus to prove that the step size you specified for steepest descent actually minimizes $\psi(x+sr)$ over all possible step sizes.

c. Suppose now that $\psi(x)$ is a general functional (not necessarily quadratic). What is Newton's method for finding the zeros of the gradient of $\psi(x)$?

d. Suppose ψ (a general functional) attains a local minimum at $x = \alpha$. Assume what you need about ψ and use a result about fixed point methods to establish local convergence of Newton's method about α.

e. How well does Newton's method work for the quadratic functional in (a)?

Numerical quadrature:

2. If we use the trapezoidal rule to evaluate $\int \frac{dx}{2+\cos x}$ over the intervals $[0,1]$, $[0,\pi]$ and $[-\pi,\pi]$, we will find that the error decreases with the step size h like $O(h^2)$, $O(h^4)$, and $O(e^{-c/h})$ respectively. Explain why we get these three different rates, and give also the value for the constant c in the particular case above.

Hint: You can assume it known that $\int_{-\pi}^{\pi}\frac{\cos nx\ dx}{2+\cos x}=\frac{2\pi}{\sqrt{3}}(\sqrt{3}-2)^n,\ n=0,1,2,\ldots$

Interpolation / Approximation:

3. a. The Chebyshev polynomials of the second kind are defined as

$$U_n(x)=\frac{1}{n+1}T'_{n+1}(x),\ n\geq 0$$

where $T_{n+1}(x)$ is the Chebyshev polynomial of degree $n+1$. Show that

$$\int_{-1}^{1}\sqrt{1-x^2}\ U_n(x)\,U_m(x)\,dx=\frac{\pi}{2}\delta_{mn}.$$

Hint: The Chebyshev polynomials satisfy the ODE $(1-x^2)T''_n-xT'_n+n^2T_n=0$ and the orthonormality relation $\int_{-1}^{1}\frac{T_n(x)\,T_m(x)}{\sqrt{1-x^2}}dx=\begin{cases}0 & \text{if } m\neq n\\ \pi & \text{if } m=n=0\\ \pi/2 & \text{if } m=n>0\end{cases}$.

b. Let $\{\phi_n(x)\}_{n=0}^{\infty}$ be a family of orthonormal polynomials with respect to the weight function $w(x)$. Show that the n:th degree polynomial that minimizes $\int_{-1}^{1}w(x)\,(f(x)-p_n(x))^2\,dx$ is given by

$$p_n(x)=\sum_{k=0}^{n}\left(\int_{-1}^{1}w(x)f(x)\,\phi_k(x)dx\right)\phi_k(x).$$

c. Show that the best degree $2n$ polynomial approximation to $\frac{1}{\sqrt{1-x^2}}$ in the norm

$$||f|| \equiv \sqrt{\int_{-1}^{1} \sqrt{1-x^2}\, f(x)^2\, dx}$$

is given by

$$\frac{1}{\sqrt{1-x^2}} \approx \frac{4}{\pi} \sum_{k=0}^{n} \tfrac{1}{2k+1} U_{2k}(x) \ . \qquad (1)$$

d. What is the rate of convergence of (1) at $x = 0$?

Hint: $\frac{d}{dx}(\arccos(x)) = -1/\sqrt{1-x^2}$.

Linear algebra:

4. Let A be in $R^{m\times n}$:

a. For general m,n, what are the three possibilities for the number of solutions of $Ax = b$? Do the possibilities change if you know $m > n$? $m < n$?

b. Define the singular value decomposition (SVD) of A and show where the rank of A comes in.

c. For general m,n, how does b relate to the left singular vectors & values of A for the case that $Ax = b$ is solvable?

d. How would you use the SVD directly to find the minimal-norm least-squares solution of $Ax = b$ (i.e. the minimizer of $||Ax - b||^2$ of minimal norm $||x||$)?

Numerical ODE:

5. The scheme for an M-stage implicit Runge-Kutta method for solving the ODE $u' = f(t, u)$ is given by

$$\begin{aligned} \xi_i &= u_n + h \sum_{j=1}^{M} a_{ij} f(t_n + c_j h, \xi_j), \ i = 1, \dots, M \\ u_{n+1} &= u_n + h \sum_{j=1}^{M} b_j f(t_n + c_j h, \xi_j) \end{aligned} \qquad (2)$$

where a_{ij}, b_i, and c_i, $i,j = 1, \dots, M$ are some constants.

a. Apply a 2-stage implicit Runge-Kutta method to solve the scalar linear ODE $u' = \lambda u$ and show that there exists a function $r(z)$ such that this scheme can be written as

$$u_n = r(\lambda h)^n u_0.$$

Hint: You may express your result in terms of the matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, and the vectors $\mathbf{1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$.

b. The 2-stage implicit Runge-Kutta method derived via collocation using Gauss-Legendre nodes leads to the amplification function

$$r(z) = \frac{1 + \frac{z}{2} + \frac{z^2}{12}}{1 - \frac{z}{2} + \frac{z^2}{12}}.$$

Show that this corresponds to an A-stable method, that is $|r(z)| \leq 1$ for z such that $\mathrm{Re}(z) \leq 0$.

Numerical PDE:

6. Consider advancing $u_t + u_x = 0$ numerically in time (on an infinite interval in space) with each of the finite difference stencils as sketched in the left column of the table below (chosen with weights that give highest possible order for the *x*- and *t*-derivatives separately, i.e. not combined as for ex. in the Lax-Wendroff scheme). One can then use either von Neumann analysis to get exact stability conditions on *k/h* (time step divided by space step) or use the CFL condition to get upper estimates for these ratios. Some entries of the comparison table are already filled in. Fill in the rest of the table (and, like for all other of the problems, show your work that underlies your answers)

Stencil	By von Neumann analysis		By CFL condition	
	stable	unstable	stable	unstable
× │ × − ×	$k/h \le 1$	$k/h > 1$	can't tell	$k/h > 1$
× │ × − ×				
× │ × − × − ×	never	always	can't tell	$k/h > 1$

× − × − ×
     │
     ×				