Department of Applied Mathematics
Preliminary Examination in Numerical Analysis
Monday August 20, 2007 (10 am - 1 pm)

Submit solutions to four (and no more) of the following six problems. Justify all your answers.

Root finding:

1. Consider the respective scalar and vector functionals

$$F(x) = \frac{a}{2} x^2 - x\, b$$

and

$$G(x,y) = \frac{1}{2} \left\langle \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \right\rangle - \left\langle \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \end{pmatrix} \right\rangle .$$

a. Determine necessary and sufficient conditions for $F(x)$ to have a unique minimizer (value of x that minimizes $F(x)$).

b. Under these conditions, write down a specific form of Newton's method for finding this minimizer (start from the general form, then simplify as much as possible, showing your work along the way), argue that the method is well posed, and discuss convergence.

c. Assuming that $G(x,y)$ has a unique minimizer (it does!), write down the specific form of Newton's method for finding this minimizer (start again from the general form, then simplify as much as possible, showing your work along the way), argue that the method is well posed, and discuss convergence.

Numerical quadrature:

2. a. Given N distinct positive nodes x_k, $k = 1,...,N$, show that there exists N weights w_k, $k = 1,...,N$, such that

$$\int_0^\infty w(x)\, r(x)\, dx = \sum_{k=1}^N w_k\, r(x_k)$$

is $N-1$ order accurate.

b. Show that if the nodes x_k are chosen as roots of the Laguerre polynomials, then there exists weights, w_k, such that

$$\int_0^\infty e^{-x} f(x)\, dx = \sum_{k=1}^N w_k\, f(x_k)$$

is $2N-1$ order accurate.

Hint: The Laguerre polynomials $L_k(x)$ are orthogonal with respect to the weight $w(x) = e^{-x}$ on the interval $[0, \infty)$. You may also use the result that, if $p(x)$ is a polynomial of degree up to $2N-1$, then there exists polynomials $s(x)$ and $r(x)$ of degree N-1 such that $p(x) = s(x)\, L_N(x) + r(x)$.

Interpolation / Approximation:

3. a. Define the minimax error of approximating a function by a polynomial.

   b. Find the maximum norm of the error of approximating e^x using the first degree Taylor polynomial on the interval [-1,1], expanding about $x = 0$.

   c. In terms of the maximum norm of the error, a least squares approximation is usually superior to a Taylor approximation. Find the first degree polynomial least squares approximation that minimizes the error of approximating e^x in the norm $\|f - g\| := \sqrt{\int_{-1}^{1} |f(x) - g(x)|^2 \, dx}$.

   d. An even better approach for approximating e^x on [-1,1] using polynomials is to use Chebyshev polynomials. Using the formula

$$e^{z\left(\frac{x+1/x}{2}\right)} = J_0(iz) + \sum_{n=1}^{\infty} (-i)^n J_n(iz)\,(x^n + x^{-n}) \,,$$

   where $J_n(z)$ denotes the nth order Bessel function, derive the expression

$$e^x = J_0(i) + 2 \sum_{n=1}^{\infty} (-i)^n J_n(i)\, T_n(x).$$

   Hint: Use the substitution $x = e^{iy}$ and remember that $T_n(x) = \cos(n \arccos(x))$.

Linear algebra:

4. Let A be an invertible $n \times n$ matrix and let $\|\cdot\|$ denote the Euclidean norm.

   a. Define cond(A), the condition number of A (with respect to solving linear systems - not for the eigenvalue problem).

   b. Suppose that you do not solve $Ax = b$ exactly, but instead obtain an approximation, $\tilde{x}$, with nonzero residual $r = b - A\tilde{x}$. Derive an upper *and* lower bound for the error in $\tilde{x}$ *relative to* $\|x\|$ in terms of cond(A), $\|r\|$, and $\|b\|$.

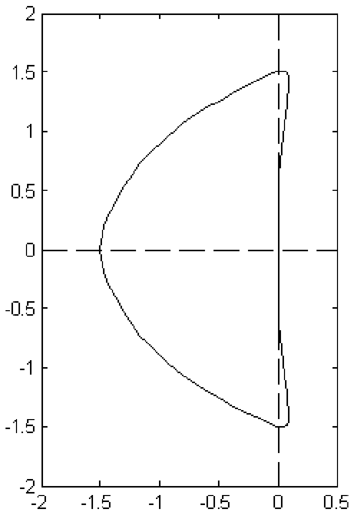
   c. Suppose now that A is orthogonal. What is $\|A\|$? Prove your claim. What is cond(A)? Prove your claim.

Numerical ODE:

5. The following is an example of a predictor-corrector scheme for solving $y' = f(t, y)$ which is not derived from the Adams Bashforth / Moulton formulas:

$$y^*_{n+1} = y_{n-1} + 2\,k\,f_n \qquad \text{(leapfrog predictor)}$$
$$y_{n+1} = \tfrac{4}{5}\,y_n + \tfrac{1}{5}\,y_{n-1} + \tfrac{2\,k}{5}(f^*_{n+1} + 2\,f_n) \qquad \text{(corrector)}$$

The stability domain of this scheme is illustrated in the figure to the right:



a. Determine the order of accuracy and check the root condition for stability.

Hints: You may use the result
$order_{\text{predictor/corrector}} = \min\,(\,order_{\text{corrector}}\,,\,order_{\text{predictor}} + 1)$

To check the root condition, it suffices to apply the scheme to $y' = 0$.

b. Show that the stability domain extends along the imaginary axis and negative real axis *exactly* to $\pm 1.5\,i$ and to -1.5 respectively.

Numerical PDE:

6. Saul'ev's scheme for approximating the heat equation, $u_t = \sigma\,u_{xx}$ (with $\sigma > 0$), consists of repeatedly using the pair of approximations

$$u(x, t + k) - u(x, t) = \tfrac{\sigma\,k}{h^2}\,(u(x + h, t) - u(x, t) - u(x, t + k) + u(x - h, t + k))$$

and

$$u(x, t + 2k) - u(x, t + k) = \tfrac{\sigma\,k}{h^2}\,(u(x + h, t + 2k) - u(x, t + 2k) - u(x, t + k) + u(x - h, t + k))$$

to advance two time levels. If we are given boundary conditions on both sides, both of the schemes are effectively explicit. However, it can be shown that the combined scheme is consistent with the heat equation only if $\frac{k}{h} \to 0$ as the computational mesh is refined.

Use von Neumann analysis to determine the stability condition on $\lambda = \sigma \frac{k}{h^2}$ if we were to

a. use just one of the two versions repeatedly.

b. use the two versions alternatingly (as is prescribed in the method).