

DEPARTMENT OF APPLIED MATHEMATICS
PRELIMINARY EXAMINATION IN NUMERICAL ANALYSIS
AUGUST, 2008

Submit solutions to four (and no more) of the following six problems. Justify all your answers.

1. Root Finding. Consider the fixed-point iteration $\mathbf{x}^{(k+1)} = M\mathbf{x}^{(k)} + \mathbf{y}$ for solving $\mathbf{x} = M\mathbf{x} + \mathbf{y}$, where $M \in \mathfrak{R}^{n \times n}$ is symmetric and $\mathbf{y} \in \mathfrak{R}^n$ is given.

(a) Show that the iteration converges for any $\mathbf{x}^{(0)} \in \mathfrak{R}^n$ if and only if the Euclidean norm of M , $\|M\|$, is less than 1.

(b) Let $A \in \mathfrak{R}^{n \times n}$ be symmetric and positive definite and $\mathbf{b} \in \mathfrak{R}^n$. Use (a) to show that the iteration $\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} - (1/\|A\|)(A\mathbf{x}^{(k)} - \mathbf{b})$ converges in the Euclidean norm to the solution of $A\mathbf{x} = \mathbf{b}$ for any $\mathbf{x}^{(0)} \in \mathfrak{R}^n$. Bound the convergence factor in terms of the eigenvalues of A .

2. Numerical quadrature. Asymptotic expansion of the trapezoidal rule (the Euler-Maclaurin formula) has the form,

$$T(h) = \int_a^b f(x)dx + \sum_{j=1}^{p-1} c_{2j}h^{2j} \left(f^{(2j-1)}(b) - f^{(2j-1)}(a) \right) + \mathcal{O}(h^{2p}), \quad p \geq 1$$

where

$$T(h) = h\left(\frac{1}{2}f_0 + f_1 + \cdots + f_{n-1} + \frac{1}{2}f_n\right),$$

and $f_i = f(a + \frac{b-a}{n}i)$, $i = 0, \dots, n$.

(a) Derive c_2 and the corresponding term of this expansion. (Hint: use integration by parts. Note: a simple statement of the value of c_2 will not count.)

Make a brief, concise statement as an answer to:

(b) How does Romberg's integration use the Euler-Maclaurin formula?

(c) What is a class of functions for which the trapezoidal rule is exact (provided that the number of nodes is sufficiently large)?

3. Interpolation/Approximation. Let the entries of $\mathbf{x} = [x_0, x_1, \dots, x_{N-1}]^T$ be N discrete samples of a continuous function f , observed at time points $t_k = 2\pi k/N$, $k = 0, 1, \dots, N-1$.

(a) What is the connection between the trigonometric interpolation of f at (t_k, x_k) , $k = 0, 1, \dots, N-1$, and the discrete Fourier transform $\mathbf{X} = F_N\mathbf{x}$, where $F_N = [f_{pq}]$ is the Vandermonde matrix with

$$\begin{aligned} f_{pq} &= \omega_N^{pq} \\ \omega_N &= e^{-2\pi i/N}. \end{aligned}$$

(b) Using $N = 16$, explain how the radix-2 splitting idea can be used to speed up the discrete transform, i.e., the calculation of the product $y = F_{16}x$.

4. Linear Algebra.

(a) Define the singular value decomposition (SVD) and Jordan canonical form of any real square matrix, $A \in \mathfrak{R}^{n \times n}$, and show that they can both be written as decompositions of the form $A = \text{matrix} \times \text{matrix} \times \text{matrix}$.

(b) How can you compute the singular values and right and left singular vectors by computing eigenvalues and eigenvectors of two operators related to A ?

(c) State *necessary and sufficient conditions* on A for it to have a decomposition that serves as *both* its SVD and its Jordan canonical form. Prove your claim.

5. Numerical ODEs. Consider a first order system of ODEs

$$\mathbf{y}' = \mathbf{f}(t, \mathbf{y})$$

with the initial condition $\mathbf{y}(0) = \mathbf{y}_0$. Consider

$$\mathbf{f}(t, \mathbf{y}) = A\mathbf{y},$$

where

$$A = \begin{pmatrix} -\frac{1000001}{2000} & -\frac{999999}{2000} \\ -\frac{999999}{2000} & -\frac{1000001}{2000} \end{pmatrix}.$$

(a) Select a first order method for solving this system and explain your choice. What accuracy do you expect if the step size $h = 10^{-4}$ on $[0, 1]$? Does this choice of step size assure stability?

Consider the implicit midpoint rule,

$$\mathbf{y}_{n+1} = \mathbf{y}_n + h\mathbf{f}\left(t_n + \frac{h}{2}, \frac{1}{2}(\mathbf{y}_n + \mathbf{y}_{n+1})\right).$$

(b) Find the order of this scheme and derive its region of absolute stability.

(c) How would you argue that the method is convergent? Is it appropriate for the linear system above?

6. Numerical PDEs. Consider the partial differential equation defined in the (t, x) -domain

$$\begin{aligned} u_{tt} &= 2u_{xx} \\ -1 &\leq x \leq 1 \\ 0 &< t \end{aligned}$$

with boundary and initial conditions

$$\begin{aligned} u(t, -1) &= u(t, 1) = 0 \quad ; \quad t > 0 \\ u(0, x) &= e^{-x^2} - e^{-1} \quad ; \quad -1 \leq x \leq 1 \\ u_t(0, x) &= (x+1)(x-1) \quad ; \quad -1 \leq x \leq 1 \end{aligned}$$

and spatial discretization with step size h and time discretization with step size k .

(a) Create an explicit $\mathcal{O}(h^2 + k^2)$ finite difference approximation to the solution.

(b) How would one accurately compute the solution at the first time point $t_1 = k$?

(c) How would one choose the sizes of h and k so as to maintain stability? Why?