

Preliminary Examination in Numerical Analysis

Department of Applied Mathematics

Tuesday, January 13, 2009 (10 am - 1 pm)

Submit solutions to **four** (and no more) of the following six problems. Justify all your answers.

1. Root Finding.

- Describe Newton's method for finding a root of $f : \mathbb{R} \rightarrow \mathbb{R}$.
- State and prove a theorem about quadratic convergence of the method. Be as general as you can and include whatever assumptions you need on the derivatives of f and the initial guess.

2. Numerical Quadrature.

- Derive Simpson's rule and its error formula for approximating $\int_a^b f(x)dx$ using the values of $f : \mathbb{R} \rightarrow \mathbb{R}$ at $a, (a+b)/2$, and b .
- Derive the composite Simpson rule and its error formula for approximating $\int_a^b f(x)dx$ using the values of f at $a + jh, j = 0, 1, \dots, n$, where $h = (b-a)/n$.

3. Interpolation/Approximation.

- Obtain the minimax first-degree polynomial approximation to $f(x) = \frac{1}{1+x}$ on $[0, 1]$.
- Formulate the theorem describing properties of the minimax error.

4. Linear Algebra.

- Define the concept of a *vector norm* on \mathbb{R}^n .
- Does $\|x\| = \sup_{p \geq 1} (\sum_{k=1}^n |x_k|^p)^{\frac{1}{p}}$ define a vector norm on \mathbb{R}^n ? (You may use the fact that $\|x\|_p = (\sum_{k=1}^n |x_k|^p)^{\frac{1}{p}}$ is a norm.)
- Does $\|x\| = \lim_{p \rightarrow \infty} (\sum_{k=1}^n |x_k|^p)^{\frac{1}{p}}$ define a vector norm on \mathbb{R}^n ?

5. Numerical ODE's.

Consider the two step method (Adams-Bashforth)

$$y_{n+2} = y_{n+1} + h \left[\frac{3}{2} f(t_{n+1}, y_{n+1}) - \frac{1}{2} f(t_n, y_n) \right].$$

Show that it is convergent, find its order, and sketch its region of absolute stability. Make sure you state relevant theorems.

6. Numerical PDE's.

Consider the heat equation

$$\frac{\partial \phi}{\partial t} = \frac{\partial}{\partial x} \left(a(x) \frac{\partial \phi}{\partial x} \right),$$

with initial condition $\phi|_{t=0} = \phi_0$ and periodic boundary conditions on the interval $[0, 1]$. Fully describe the Crank- Nicolson scheme for this problem, using a staggered grid for the spatial operator. Taking $a(x) = 1$, show that the scheme is unconditionally stable.