PRELIMINARY EXAMINATION
PARTIAL DIFFERENTIAL EQUATIONS
January 11th, 1996

Solve any **four** (4) of the following five (5) problems. Only four problems will be graded—you must state which of the problems you wish to have scored. You may use the sheet of formulae provided and a calculator. No other books or tables can be used. Good luck!

1. Given

(A)
$$\begin{aligned} x^2u_t - 2xu_x &= -2u \quad , \ x \in I\!R \ , \ t \geq 0 \\ u(x,0) &= x^2 \end{aligned}$$

and

(B)
$$\begin{aligned} u_t + u^2u_x &= 0 \quad , \ x \in I\!R \ , \ t \geq 0 \\ u(x,0) &= x \end{aligned}$$

answer the following for both (A) and (B).

(a) What are the characteristic equations?

(b) Find the characteristics and the solution along each characteristic.

(c) Find $u(x,t)$ if possible. If no solution, explain why.

(d) If solution exists only up to some finite time t, find t.

2. (a) Find a formal series solution of the problem

$$(H)\begin{cases} u_{tt} - u_{xx} + u = 0 \ , & 0 < x < \pi, \ t > 0 \\ u(0,t) = 0 = u(\pi,t) \ , & t > 0 \\ u(x,0) = 0, \quad u_t(x,0) = h(x) \ , & 0 < x < \pi \end{cases}$$

Define all constants in your solution in terms of the Fourier coefficients of h.

(b) Give some reasonable condition on h so that your series solution in part (a) is a classical solution of (H).

(c) If possible, find $\lim_{t\to\infty} |u(\pi/2,t)|$.

3. Let $\phi(x,y,z)$ solve the Poisson equation in 3 dimensions:

$\Delta\phi = G(x,y,z)$ in a domain D

$\phi = 0$ on the boundary ∂D,

where $G(x,y,z)$ is a known function that is continuous on D. Prove that the problem has at most one solution. (If you use the maximum principle to establish your result, you must first prove it.)

The following variation of the heat equation is a simplified version of a problem from David Sholl's PhD thesis.

$$\boxed{\text{P}}\begin{cases} \partial_t u = \partial_x^2 u, & 0 < x < L, \quad t > 0, \\ \partial_t u = \partial_x u - u, & x = 0, \quad t > 0, \\ \partial_x u = 0, & x = L, \quad t > 0, \\ u(x,0) = U(x), & 0 < x < L, \quad t = 0. \end{cases}$$

The next two problems are based on $\boxed{\text{P}}$. The two problems are independent; neither one requires knowledge about the other.

4. (a) Does $\boxed{\text{P}}$ have a steady state? If so, what is it?

   (b) Using separation of variables, find a countably infinite set of real-valued solutions of the PDE and boundary conditions. (The specification of these solutions involves a transcendental equation that cannot be solved exactly. If is sufficient to show that the equation has infinitely many roots, each of which corresponds to a solution of the PDE.)

   (c) Let m_n denote the n^{th} eigenvalue for $\boxed{\text{P}}$. Show that as $n \to \infty$, $m_n \to (n - 3/2)\pi/L$.

5. $\boxed{\text{P}}$ has a natural "energy":

$$E(t) = \int_0^L [u(x,t)]^2 dx + [u(0,t)]^2.$$

   (a) Show that if u satisfies $\boxed{\text{P}}$, then $\dfrac{dE}{dt} \leq 0$. For what class of functions $\{u(x,t)\}$ does this argument apply?

   (b) Prove uniqueness for $\boxed{\text{P}}$. Within what class of functions is the solution unique?