PDE Preliminary exam January 9, 1997

Solve any **four** (4) of the following five (5) problems. You must state which if the problems you wish to have scored. You may use the sheet of formulae provided and a calculator, but no other books or tables.

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1) In this problem $f(t)$, $g(x)$ are smooth infinitely differentiable functions and $g(x)$ is a function which decays to zero for large $|x|$.

a) Solve the following equation with the boundary condition given:

$$U_t + U_x = f(t), \quad -\infty < x < \infty, \ -\infty < t < \infty$$
$$U(x = 0, t) = 0$$

What is the main qualitative behavior of the solution?

b) Suppose we wish to supplement the above with the initial condition: $U(x,t=0)=g(x)$. What can be said in this case?

c) Solve the following equation

$$U_t + UU_x = f(t), \quad -\infty < x < \infty, \ -\infty < t < \infty$$
$$U(x, t = 0) = g(x)$$

How do the qualitative properties of the solution of this equation differ from that of part (a)?

2) Let $f(x)$ be a square integrable function on the interval $[0,L]$. Let I_N be the integral

$$I_N = \int_0^L dx \left(f(x) - \sum_{n=1}^{N} b_n \sin(n\pi x/L) \right)^2$$

a) Show that I_N is minimized for any N if the b_n are given by the Fourier sine coefficients of $f(x)$.

b) Derive Bessel's inequality for $\sum_{n=1}^{\infty} b_n^2$ using I_N.

c) State a condition under which this inequality becomes an equality (Parseval's relation). You do not need to prove this result.

d) Let $L=\pi$, and $f(x) = x$. Assuming Parseval's relation is correct, use it to obtain a series representation for π^2.

3) Suppose $u(x,y)$ is a harmonic function in a bounded 2 dimensional domain D that includes the origin. We are going to examine the local behavior of u near the origin. Represent u as a double power series:

$$u(x,y) = \sum_{m,n=1}^{\infty} a_{mn} x^m y^n \qquad (*)$$

a) Suppose (0,0) is a critical point of u. What does this imply about the power series coefficients a_{mn}?

b) Suppose u is harmonic, $\Delta u = 0$. What restrictions does this place on the coefficients a_{mn} for m,n = 0, 1, 2? (*You need not look at higher powers*)

c) Is the critical point (0,0) a maximum, minimum or neither?

d) State a general result that explains the result in (c). Under what conditions is your "theorem" valid? When does (*) satisfy these conditions?

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4) Consider the "Green's function" $G(x,t)$ satisfying

$$G_t = G_{xx}, \qquad -\infty < x < \infty, \; t > 0$$
$$G(x, t = 0) = \delta(x)$$

where $\delta(x)$ is the Dirac delta "function," which is defined by the property

$$\int_{x-\varepsilon}^{x+\varepsilon} dx \; f(x')\delta(x-x')dx' = f(x), \; \varepsilon > 0$$

a) Use Fourier transforms to establish that $G(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-k^2t}\, dk$

b) Show that the above integral can be evaluated in closed form and find

$$G(x,t) = \frac{1}{\sqrt{4\pi t}} \exp(-x^2/4t)$$

c) Use the Green's function above to solve the equation

$$U_t = \kappa U_{xx} \quad -\infty < x < \infty, \quad t > 0$$
$$U(x,0) = h(x)$$

Hint: Note that $h(x) = \int_{-\infty}^{\infty} dx'\, h(x')\, \delta(x-x')$.

d) Suppose $h(x)$ has compact support, e.g. $h(x) = 0$ if $|x| > L$. Thus $u(2L,0) = 0$. Find the smallest time for which $u(2L,t) \neq 0$.

5) The wave equation describes many physical phenomena. One case is that of sound waves, in which the "displacement," $p(x,t)$, represents the deviation of the pressure of the gas from atmospheric pressure. Suppose you have an organ pipe, closed at one end, and open at the other. The appropriate boundary conditions are that the pressure is held constant, at its atmospheric pressure at the closed end, and that the gradient of the pressure normal to the open boundary is zero at the open end. Denote the speed of sound in air by c_S.

a) Formulate the one dimensional wave equation for this system, with appropriate boundary and initial conditions (*You do not need to derive the wave equation*).

b) Obtain a formal solution of the problem.

c) Under what conditions on the boundary and initial conditions is your formal solution a "classical solution" of the wave equation (e.g. smooth enough to satisfy the equation).

d) If the speed of sound is 330 m/s, and the organ pipe is 3 m long, what is the lowest frequency at which the pipe will normally oscillate?