Preliminary Exam Partial Differential Equations January 1998

There are 5 problems. You must do 4 of them. Please mark which four you choose—only four problems will be graded.

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1) (25 pts) Let $u(x,y)$ satisfy Laplace's equation in a semi-infinite strip:

$$\Delta u = 0 \qquad 0 < x < \infty, \quad 0 < y < H$$
$$u(0,y) = 1 \quad 0 < y < H$$
$$u(x,0) = \begin{cases} 1 & 0 < x < L \\ 0 & x > L \end{cases}$$
$$u(x,H) = 0 \quad 0 < x < \infty$$
$$u(x,y) \text{ bounded as } x \to \infty, \ 0 < y < H$$

a) Obtain a formal solution to this equation.

b) Evaluate $u(L,H)$.

c) Evaluate $u_x(L,H)$

d) Does u have a finite limit as $x \to \infty$? If so, what is it? Justify your answer.

2) (25 Pts)

a) The following two statements are apparently in conflict (a most amusing paradox). Why is this a paradox? Explain how the first statement can be modified to eliminate the paradox.

i) "If two 2π periodic, L_2 functions have identical Fourier series then they are equal."

ii) "The functions $f(x) = \ln(\cosh x)$ and $g(x) = \begin{cases} 1 & x = 0 \\ \ln(\cosh x) & x \neq 0 \end{cases}$ have the same Fourier series on the interval $-\pi < x < \pi$."

b) Assume the following

Theorem (Convergence in the Mean): Let f be continuous and periodic on $\mathbb{R}$ with period $2L$, then the Fourier series for f converges to f in the L_2 sense, i.e. if $S(x)$ is the Fourier series for f, then $\int_{-L}^{L} \left(f(x) - S(x) \right)^2 dx = 0$

Prove the proper statement of (i).

c) Derive Parseval's theorem, $\frac{1}{L}\int_{-L}^{L} |f(x)|^2 dx = \frac{a_0^2}{2} + \sum_1^\infty |a_n|^2 + |b_n|^2$, using the convergence theorem in (b).

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3) (25 pts) The following nonlinear PDE is a special case of an equation known as the *Monge-Ampère* equation. In this problem we reduce the equation to an equivalent first-order equation and solve it.

$$\phi_{tt}\phi_x^2 - 2\phi_{xt}\phi_t\phi_x + \phi_t^2\phi_{xx} = 0 \qquad \text{(I)}$$

Here subscripts denote partial derivatives, as usual.

a) Show that (I) is equivalent to

$$\frac{\phi_{tt}}{\phi_x} - \frac{\phi_t\phi_{xt}}{\phi_x^2} = \frac{\phi_t\phi_{xt}}{\phi_x^2} - \frac{\phi_t^2\phi_{xx}}{\phi_x^3} \qquad \text{(II)}$$

[*Hint: this part is easy*.]

b) Show that (II) can be written as a first order equation for $u = \phi_t/\phi_x$. What is the first order equation? [*Hint: we ordered the terms in (II) as they are for a reason*!]

c) Suppose that we are given the initial conditions

$$\phi(x,0) = 1 + 2e^{3x}$$
$$\phi_t(x,0) = 4e^{3x}$$

Find $u(x,t)$ for $t > 0$. Then find $\phi(x,t)$ for $t > 0$.

For Problems (4) and (5): The transverse vibrations of string with no stiffness are governed by the wave equation. When stiffness is important, an additional term proportional to u_{xxxx}, must be added, to give the equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} - a^2 \frac{\partial^4 u}{\partial x^4} \qquad \text{(III)}$$

To this equation we add:

$$\textit{Boundary Conditions:} \quad \begin{cases} u(0,t) = 0 \\ u(L,t) = 0 \\ \dfrac{\partial^2 u}{\partial x^2}(0,t) = 0 \\ \dfrac{\partial^2 u}{\partial x^2}(L,t) = 0 \end{cases} \qquad \textit{Initial Conditions:} \quad \left.\begin{cases} u(x,0) = U(x) \\ \dfrac{\partial u}{\partial t}(x,0) = V(x) \end{cases}\right\} \; 0 < x < L \qquad \text{(IV)}$$

4) (25 Pts) Prove uniqueness for this model. More precisely, prove that it has at most one solution that is twice-continuously differentiable in x ($C^2[0,L]$) and that evolves from the initial data. What assumptions are required on the functions U and V?

5) (25 Pts)

a) Find the formal solution of (III) with (IV). Be sure that all symbols in your solution are defined clearly.

b) Define the <u>fundamental period</u> of the motion to be the earliest time after t=0 at which the lowest Fourier mode returns to its initial position and velocity. (If the lowest mode is constant in time, then the fundamental period is the time at which the next lowest Fourier mode returns to its initial position and velocity). Denote this period by T. What is T?

c) Define the <u>fundamental frequency</u> by $\omega(c, a) = 1/T$. For $a^2>0$ is this frequency higher or lower than that for $a^2 = 0$? In other words, does the stiffness of the string raise or lower the fundamental frequency?

d) Is the motion of the string periodic if $a^2 = 0$? If so, find the period of the motion. If not, explain why. (Assume that $U(x)$ and $V(x)$ are such that all Fourier modes are nonzero).

e) Is the motion of the string periodic if $a^2 > 0$? If so, find the period of the motion. If not, explain why. (Assume that $U(x)$ and $V(x)$ are such that all Fourier modes are nonzero).