Preliminary Exam Partial Differential Equations August 1998

There are 5 problems. You must do 4 of them. Please mark which four you choose—only four problems will be graded.

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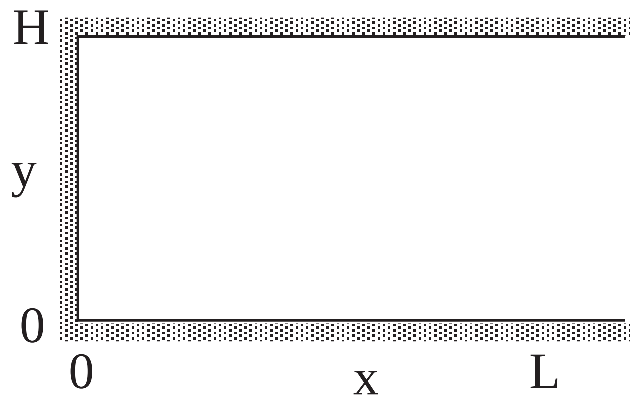
1) (25 Pts) Let $u(x,y)$ satisfy Laplace's equation in a semi-infinite strip:

$$\Delta u = 0 \qquad 0 < x < \infty, \quad 0 < y < H$$
$$u(0,y) = 1 \quad 0 < y < H$$
$$u(x,0) = \begin{cases} 1 & 0 < x < L \\ 0 & x > L \end{cases}$$
$$u(x,H) = 0, \quad 0 < x < \infty$$
$$u(x,y) \textit{ bounded as } x \to \infty, \quad 0 < y < H$$



a) Obtain a formal solution to this equation.

b) Evaluate $u(L,H)$.

c) Evaluate $u_X(L,H)$

d) Does u have a finite limit as $x \to \infty$? If so, what is it? Justify your answer.

2) (25 Pts) The objective of this problem is to solve:

$$\partial_t u + x\partial_x u = \nu \partial_x^2 u \quad , \quad -\infty < x < \infty \ , \ \nu \geq 0$$
$$u(x,0) = \sin(kx)$$

The problem is not straightforward, but it can be solved with a combination of characteristics and separation of variables.

a) For $\nu = 0$, the problem can be solved by the method of characteristics. Find both the characteristics and the solution of the problem in this special case.

b) For $\nu > 0$, change variables: $(x, t) \to (z, \tau)$, where $\tau = t$, and $z(x, t)$ is constant along a characteristic found in part (a), and $z(x, 0) = x$. Show that the problem can be solved by separation of variables in these new coordinates.

c) Find the complete solution of the problem in terms of (z, τ). Rewrite this to find $u(x, t)$.

d) Show that as $\nu \to 0$, the solution in (c) reduces to that in (a).

e) The limiting behavior of the solution as $t \to \infty$ is delicate.

(i) Find the limiting behavior as $\tau \to \infty$, with z fixed.

(ii) Find the limiting behavior as $t \to \infty$, with x fixed.

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3) (25 Pts) Let $f(x)$, $-L < x < L$, be a piecewise smooth function. Let a_n and b_n be the Fourier coefficients for f and α_n and β_n be the Fourier coefficients for f'.

a) Prove that there is a $c > 0$ such that $|a_n| \le c/n$ for all $n > 0$ (this shows that $a_n = O(1/n)$).

For the next <u>two</u> parts, we make the additional assumption that f is continuous and $f(-L + 0) = f(L - 0)$.

b) Show that

$$\alpha_n = \frac{n\pi}{L} b_n \quad , \quad \beta_n = -\frac{n\pi}{L} a_n$$

c) Use (b) to prove that $a_n = o(1/n)$ as $n \to \infty$, i.e., $|na_n| \to 0$ as $n \to \infty$.

4) (25 Pts) Let

$$u_t - u_{xx} = 1 - x^2 \;,\; 0 \le x \le 1 \;,\; t \ge 0$$
$$u(0,t) = 0 \;,\quad u(1,t) = 0$$
$$u(x,0) = 0$$

a) State and prove an appropriate maximum principle that applies to this equation, and use it to show that $u(x,t) \ge 0$.

In the next several parts, we will use this principle to obtain an upper bound on the solution.

b) First find the equilibrium solution to the equation, $u_e(x)$.

c) Let $v(x,t) = u(x,t)-u_e(x)$. Find the equation solved by v.

d) Use the maximum principle for v to find an upper bound on the solution $u(x,t)$.

5) (25 Pts) Consider the initial value problem (Note: this equation differs from #4 in that there is a 2nd derivative in time!)

$$u_{tt} - u_{xx} = 1 - x^2, \quad 0 \le x \le 1 \quad t \ge 0$$
$$u(0,t) = u(1,t) = 0$$
$$u(x,0) = u_t(x,0) = 0$$

a) Find the equilibrium solution to this equation, i.e. a solution, u_e, that does not depend upon time.

b) State the initial value problem for the function $v(x,t) = u(x,t) - u_e(x)$.

c) Find the minimum and maximum values of $u(x,t)$ for all $t \ge 0$.