

There are 5 problems. You must do 4 of them. Please mark which four you choose—only four problems will be graded.

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1)(25 Pts) Let  $u(x, t)$  be defined by the formal integral,

$$u(x, t) = \frac{1}{2} \int_{-\infty}^{\infty} (m) e^{-m^2 t} e^{imx} dm$$

where  $(m)$  is bounded and continuous for all  $m$ .

*Comment: If you use any parts of the Riemann-Lebesgue lemma in your arguments below, you must prove them.*

- By formally differentiating under the integral sign, show that  $u(x, t)$  satisfies the heat equation,  $u_t = u_{xx}$  provided the derivatives exist.
- Show that for any  $t > 0$ ,  $u$  is uniformly bounded in  $x$ . [The argument for  $u_{xx}$  is identical. This justifies the calculation in (a).]
- Show that for  $t > 0$ ,  $u(x, t) \rightarrow 0$  as  $x \rightarrow \pm\infty$ .

2)(25 Pts) The electromagnetic field in a medium obeys the wave equation with a speed  $c$  that depends upon the medium. Suppose the medium has a jump discontinuity at  $x = 0$ , with speed  $c_1$  for  $x < 0$ , and with speed  $c_2$  for  $x > 0$ :

$$\begin{aligned} E_{tt} - c_1^2 E_{xx} &= 0, & x < 0, & t > 0 \\ E_{tt} - c_2^2 E_{xx} &= 0, & x > 0, & t > 0 \end{aligned}$$

The electric field and its spatial derivative are continuous across  $x = 0$ :

$$\begin{aligned} E(0_-, t) &= E(0_+, t) \\ E_x(0_-, t) &= E_x(0_+, t) \end{aligned}$$

Suppose that the field is initially a localized pulse  $P(x)$  which is nonzero only in a domain  $0 < a < x < b$ , and that  $E_t$  is initially zero.

- Find  $E(x, t)$ . [Hint: you may find it useful to solve the problem in the two domains  $x \geq 0$ , and later impose the conditions at  $x = 0$ ]
- When does the pulse first enter the region  $x < 0$ ?
- What is the width of the pulse once it has entered the region  $x < 0$ ?

3)(25 Pts) Let  $u(x, t)$  be the temperature in a rod of length  $L$ . It satisfies the following problem.

$$\begin{aligned} t u &= k \frac{\partial^2 u}{\partial x^2}, & 0 < x < L, \quad t > 0 \\ x u &= Q & x = 0, \quad t > 0 \\ u &= 0, & x = L, \quad t > 0 \\ u(x, 0) &= 0, & 0 < x < L, \quad t = 0, \end{aligned}$$

where  $Q$  is a constant.

- What is the steady-state temperature distribution in the rod?
- What is the maximum temperature in the rod at steady state? Where does it occur?
- At steady state, what is the rate of heat-transfer to the rod?
- Find  $u(x, t)$  for  $t > 0$ .
- Approximately how long is required before the temperature in the rod is within 1% of its steady-state value at every  $x$ ? Justify your answer.

4) (25 Pts) The objective of this problem is to solve:

$$\begin{aligned} \frac{1}{y} \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} &= u^2 \\ u(x, 1) &= x^2 \end{aligned}$$

- For what values of  $(x, y)$  does the existence and uniqueness theorem imply that there exists a unique solution to this problem?
- Find the solution  $u(x, y)$ .
- For what values of  $(x, y)$  does the solution exist?

5) (25 Pts) Consider the Laplace equation in the 3D sphere with “Robin”-type boundary conditions

$$\begin{aligned} u &= 0, \quad r < 1 \\ \frac{\partial u}{\partial r} + u &= f(\theta, \phi), \quad r = 1 \end{aligned}$$

where  $(r, \theta, \phi)$  are spherical coordinates and  $f > 0$  is a constant. Prove that this equation has at most one solution  $u(r, \theta, \phi)$ . Be sure to carefully state your assumptions on  $u$  and  $f$ .