PDE Preliminary exam
January 17, 2002

Solve any four (4) of the following five (5) problems. You must state which of the problems you wish to have scored. You may use the sheet of formulae provided and a calculator, but no other books or tables.

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1. Find the function $u(x,y)$ that solves

$$u_x - \frac{y^3}{2} u_y = u, \quad x > 0, \quad y \in \mathbb{R}$$
$$u(0,y) = f(y), \quad y \in \mathbb{R}$$

for a given function $f(y)$. In what region is your solution valid? Why?

2. Let H be a Hilbert space with inner product $(,)$ and $\{\phi_k\}$, $k = 0,1,2,\ldots$ be a (countable) orthogonal system of elements of H. Examples of orthogonal systems for the the space of square integrable functions on $(-1,1)$ with the usual inner product are the cosines $\phi_k = \cos k\pi x$, $k = 0,1,2,\ldots$, and the Legendre polynomials $\phi_k = \frac{1}{2^k k!}\frac{d^k}{dx^k}(x^2-1)^k$, $k = 0,1,2,\ldots$.

(i) Let $S_n = \sum_{k=0}^n a_k \phi_k$. Show that for any fixed n, and all $f \in H$, the distance $||f - S_n||^2$ is minimized by the choice $a_k = (f,\phi_k)/||\phi_k||^2$. What is the minimum of $||f - S_n||^2$?

(ii) State and prove Bessel's inequality for the orthogonal system $\{\phi_k\}$.

(iii) From now on assume that the ϕ_k also have unit norm. An orthonormal system $\{\phi_k\}$ is said to be *complete* if for any $f \in H$ and $\epsilon > 0$, there exist m, and $c_0, \ldots, c_m$ such that $||f - \sum_{k=1}^m c_k\phi_k|| < \epsilon$. Show that $\{\phi_k\}$ is complete if and only if $||f||^2 = \sum_{k=1}^\infty (f,\phi_k)^2$, $\forall f \in H$.

(iv) Show that if $\{\phi_k\}$ is complete then $(g,\phi_k) = 0$ for all k implies $g = 0$ (a.e.). Also show the converse: if $(g,\phi_k) = 0$ for all k implies $g = 0$ then $\{\phi_k\}$ is complete. *Hint:* for the converse, consider the sequence of functions $g_n = f - \sum_{k=1}^n (f,\phi_k)\phi_k$, and their limit g.

3. Suppose that $u(x,t)$ for $x \in \mathbb{R}^3$ solves

$$u_t - \triangle u = 1, \quad |x| < 1,\ t > 0$$
$$u(x,0) = 0, \quad |x| \le 1$$
$$u(x,t) = 0, \quad |x| = 1,\ t > 0\ ,$$

where $r = |x| = (x_1^2 + x_2^2 + x_3^2)^{\frac{1}{2}}$.

(i) Give a physical argument that this problem should have a steady-state, radially symmetric solution $u = S(r)$, where $r = |x|$, and that $u(x,t) \le S(r)$, for $|x| \le 1$ and $t > 0$.

(ii) Find $S(r)$ and prove that

$$u(x,t) \le S(r), \quad |x| \le 1, t > 0\ .$$

(iii) Find constants $\alpha, \beta > 0$ so that

$$u(x,t) \ge \beta\left(1 - e^{-\alpha t}\right)\frac{1-r^2}{6}\ .$$

4. Consider the initial value problem for the transverse displacement $y(x,t)$ of an elastic beam

$$\begin{aligned} y_{tt} + a^2 y_{xxxx} &= 0, \quad x \in \mathbb{R}, \quad t > 0 \\ y(x,0) &= f(x) \\ y_t(x,0) &= a g''(x) \end{aligned}$$

where $f(x)$ and $g(x)$ are functions that are as smooth as you like.

(i) Find a formal solution for $y(x,t)$ using Fourier transforms.

(ii) Suppose that you know

$$\int_{-\infty}^{\infty} e^{\pm i\chi k^2} e^{i\phi k}\, dk = (1 \pm i)\sqrt{\frac{\pi}{2\chi}} e^{\mp i \frac{\phi^2}{4\chi}},$$

where χ, ϕ are real constants and the integrals are to be understood in the proper sense. Show that your solution y in (i) can be written in the form

$$y(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Big(f(x - 2v\sqrt{at})K(v) + g(x - 2v\sqrt{at})L(v) \Big)\, dv$$

What are K and L?

5. Consider the boundary value problem

$$\triangle u = 0, \quad x^2 + y^2 > 1$$

$$u(x,y) = \begin{cases} 1, & x^2 + y^2 = 1, \quad y < 0 \\ 0, & x^2 + y^2 = 1, \quad y > 0 \end{cases}$$

and $u(x,y)$ is bounded for $x^2 + y^2 > 1$.

(i) Find an explicit representation for $u(x,y)$.

(ii) Determine upper and lower bounds for $u(x,y)$ on $x^2 + y^2 > 1$.

(iii) Evaluate $u(x,0)$ for $x > 1$.