PDE Preliminary exam
August 22, 2002

Solve any four (4) of the following five (5) problems. You must state which of the problems you wish to have scored. You may use the sheet of formulae provided and a calculator, but no other books or tables.

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1. Consider the equation

$$u_t - e^{-u} u_x = 0 \ . \tag{1}$$

a) Solve the characteristic ODEs.

b) Consider (1) with the initial condition $u(x,0) = g(x)$, where g is a continuous, positive function on $\mathbb{R}$ satisfying $g(x) \to 0$ as $|x| \to \infty$. Does the solution $u(x,t)$ obtained by the method of characteristics make sense for all $t > 0$ and all $x \in \mathbb{R}$? Explain.

c) Find an explicit solution to (1) of the form $u(x,t) = f(x/t)$. (The solution you find may be defined on some subdomain of the x, t plane.)

2. The "free" Schrödinger equation is

$$i\frac{\partial \Psi}{\partial t} + \frac{\partial^2 \Psi}{\partial x^2} = 0 \tag{2}$$

a) Find its solution on $-\infty < x < \infty$ with initial data $\Psi(x,0) = \delta(x - x')$ where $\delta(x)$ is the Dirac delta function (Hint: the key property of the Dirac delta is: $\int_{\mathbb{R}} f(x)\delta(x - x')dx = f(x')$). Write the solution in explicit form—not in terms of integrals.

Use this formulation to obtain a formal solution of eq. (2) with initial data

$$\Psi(x,0) = f(x) \tag{3}$$

b) Consider the Schrodinger equation with a potential:

$$i\frac{\partial \Psi}{\partial t} + \frac{\partial^2 \Psi}{\partial x^2} - x^2\Psi = 0 \tag{4}$$

with $\Psi(x,0) = f(x)$ on $-\infty < x < \infty$. Solve (4) with initial value (3). You may use the following results: the Sturm-Liouville problem

$$\frac{d^2\phi_n}{dx^2} + (E_n - x^2)\phi_n = 0$$

with $E_n = 2n+1$ has a complete, othonormal set of solutions: $\phi_n(x) = H_n(x)e^{-x^2/2}$, $n = 0, 1, 2 \ldots$ where $H_n(x)$ are suitably normalized polynomials.

3. Suppose that f is a continuous function on the interval $[-L, L]$.

a) State conditions under which the Fourier series representation of $f(x)$ is equal to $f(x)$ for all x in the interval (a proof is not required).

b) State conditions under which the Fourier cosine series representation of $f(x)$ on the interval $[0, L]$ is equal to $f(x)$ for all x in the interval $[0, L]$ (a proof is not required).

c) Suppose that f has two continuous derivatives on $[-L, L]$. Show that its Fourier cosine coefficients obey the bound $|a_n| < C/n^2$, giving an appropriate constant $C > 0$.

4. Consider the initial value problem for the heat equation on the line

$$u_t = Du_{xx}, \quad u(x,0) = f(x), \tag{5}$$

with $D > 0$. Recall that the heat kernel is given by $k(z,t) = (4\pi Dt)^{-\frac{1}{2}} e^{-\frac{Z^2}{4Dt}}$.

a) Show that $\int_{\mathbb{R}} u(x,t)dx$ is conserved and that $\int_{\mathbb{R}} |u(x,t)|^2 dx$ decreases in time (state appropriate sufficient conditions on f for these statements to hold).

b) Show that

$$|u(x,t)| \leq \max_{y\in\mathbb{R}} |f(y)|, \quad \text{for all } x \in \mathbb{R}, t > 0\ ,$$

and that therefore the maximum of $|u(x,t)|$ is nonincreasing in time (assume that f is bounded).

c) Consider the initial condition

$$f(x) = \begin{cases} \frac{1}{2}, & |x| \leq 1 \\ 0, & |x| > 1 \end{cases}.$$

Give an integral representation of the solution of (5) for $t > 0$, and show that as $t \to +\infty$ we have

$$|u(x,t)| \leq \frac{C}{\sqrt{Dt}}\ , \quad \text{for all } x \in \mathbb{R}\ .$$

for some C that is independent of x.

5. Consider the wave equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0 \tag{6}$$

a) Show that its "characteristics" are: $\xi = x + ct$, $\eta = x - ct$. Explain the meaning of "characteristics".

b) Use these characteristics to transform the equation and thereby find the general solution to the wave equation (6). Use this result to solve the initial value problem: $u(x,0) = f(x), u_t(x,0) = g(x)$.

c) Consider the multidimensional wave equation:

$$\nabla^2 u - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0$$

Assume radial symmetry. Find the general solution in three spatial dimensions. (Hint: consider the transformation $w = ru$, where r is the radial coordinate.)