PARTIAL DIFFERENTIAL EQUATIONS PRELIMINARY EXAMINATION
January 2006

You have three hours to complete this exam. Each problem is worth 25 points. Work only four of the five problems problems. Please mark which four that you choose—only four will be graded. Please start each problem on a new page. A sheet of convenient formulae is attached.

1. Let D be a region in $\mathbb{R}^n$, and suppose that u satisfies the equation

$$\nabla^2 u = F(x), \quad x \in D$$

   a) Suppose D is bounded and that

$$\frac{\partial u}{\partial n} = h(x), \quad x \ \ on \ \ \partial D$$

   where h given on the closed boundary ∂D, and n is the outward unit normal. Discuss the uniqueness of the solution.

   b) Consider the same problem as in (a) setting $F = 1$ and $h(x) = 1$. Discuss the existence of a solution.

   c) Now suppose that D is the exterior to the unit disc in $\mathbb{R}^2$: $D = \{x^2+y^2 > 1\}$, that $F = 1$, and that $u = 1$ on $x^2 + y^2 = 1$. Find the general solution and the bounded solution.

2. Consider the equation

$$u_{tt} - c^2 u_{xx} = H(x,t), \quad |x| < \infty \ \ t \geq 0 \ .$$

   a) Let $u(x,0) = f(x), u_t(x,0) = g(x)$ and suppose that $H = 0$. Formulate an integral, $I, I \geq 0$ that can be associated with the energy. Show I is a constant of the motion. Find a second integral, call it J, that is also a constant of motion.

   b) Let $u(x,0) = f(x) = 0, u_t(x,0) = g(x) = K =$ constant and suppose that $H = 0$. Find the solution.

   c) Suppose $H(x,t) = \delta(x)e^{i\omega t}$ where $\delta(x)$ is the Dirac delta function. Discuss possible solutions.

—over—

3. Consider

$$x^2 u_x + y^2 u_y = u\,, \quad x > 0,\ y > 0$$

a) Find and sketch the characteristics for the above equation.

b) Solve the above equation with the initial condition $u(1, y) = y^2$ for $y > 0$.

c) Does the solution exist everywhere in the first quadrant? If not, where does it fail to exist?

d) What is $\lim_{t\to\infty} u(t,t)$? What is $\lim_{(x,y)\to(0,0)} u(x,y)$?

4. (a) Find a formal series solution of the problem

$$\left.\begin{array}{ll} u_{tt} - u_{xx} - u = 0, & 0 < x < \pi,\ \ t > 0 \\ u(x,0) = 0, & 0 < x < \pi \\ u_t(x,0) = g(x), & 0 < x < \pi \\ u_x(0,t) = u_x(\pi,t) = 0, & t > 0 \end{array}\right\} \qquad (1)$$

Define all constants in your solution in terms of the Fourier coefficients of g.

(b) Give some reasonable conditions on g so that your series solution in part (a) is a classical solution of (1).

(c) Find the limiting behavior of $|u|$ at $x = \frac{\pi}{2}$ as $t \to \infty$, that is find $\lim_{t\to+\infty} \left|u\left(\frac{\pi}{2}, t\right)\right|$, if it exists.

5. Consider the Fourier series on $[0, \pi]$ given by

$$f(x) = \sum_{n=1}^{\infty} \frac{n}{1+n^2} \sin(nx)$$

a) State Parseval's relation for f.

b) Is $f \in L_2$? Is f continuous?

c) What is the Fourier series for $F(x) = \int_0^x f(\xi)d\xi$?

d) Does the Fourier series for $F(x)$ converge to a continuous function?