

PARTIAL DIFFERENTIAL EQUATIONS PRELIMINARY EXAMINATION
August 2008

You have three hours to complete this exam. Each problem is worth 25 points. Work only **four** of the five problems. Please **mark** which four you choose—only four will be graded. Please start each problem on a new page. A sheet of convenient formulae is provided.

1. Eigenvalue Problem (25 points). Solve the eigenvalue problem

$$x^2 y'' + 3xy' + \mu y = 0, \quad 1 < x < e, \quad y(1) = y(e) = 0.$$

Assume that $\mu > 1$. (Hint: Look for solutions of the form $y(x) = x^m$ and determine μ .)

2. Fourier Series (25 points).

- (a) Find the Fourier cosine series for the function

$$f(x) = x, \quad 0 \leq x \leq 1.$$

To what function on \mathbb{R} does this series converge? (Take care to specify the function for each x .)

- (b) In certain cases a Fourier series may be differentiated term-by-term to obtain the series for $f'(x)$. State a theorem to this effect.
- (c) Does this theorem apply to (a)? If so, what is the Fourier series for $f'(x)$? To what function on \mathbb{R} does this series converge?
- (d) In certain cases a Fourier series may be integrated term-by-term to obtain the Fourier series for $\int f(x)dx$. State *AND PROVE* a theorem to this effect.
- (e) Does this theorem apply to (a)? If so compute the Fourier series for $\int f(x)dx$. To what function on \mathbb{R} does this series converge?
3. Damped Wave equation (25 points). Consider the PDE

$$u_{tt} + u_t = u_{xx}, \quad 0 \leq x \leq \pi, \quad t \geq 0, \quad (1)$$

with the boundary conditions $u(0, t) = u(\pi, t) = 0$, and initial data $u(x, 0) = f(x)$, $u_t(x, 0) = 0$.

- (a) Show that any solution of (1) has the property that $\lim_{t \rightarrow \infty} u(x, t) = 0$.
- (b) Find a formal solution for $u(x, t)$.
- (c) Under what conditions on $f(x)$ is your formal solution a true solution of (1)?
- (d) What is the solution when $f(x) = \sin x$?

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4. First Order PDE (25 points). Let $\varphi(x, y, z)$ solve the PDE

$$x \frac{\partial \varphi}{\partial x} + y \frac{\partial \varphi}{\partial y} + z \frac{\partial \varphi}{\partial z} = \varphi \quad (2)$$

where $\varphi(x, y) = F(x, y)$ on the surface $z = x^2 + y$.

- (a) Find the characteristics of (2).
 - (b) Find the formal solution of (2).
 - (c) What do you need to assume about $F(x, y)$ for your solution to solve (2)?
 - (d) On what region of \mathbb{R}^3 is the solution you obtained in (b) valid? What goes wrong and why?
5. An Inhomogeneous Heat Equation (25 points).

- (a) Find a formal solution to

$$u_t = u_{xx} + xu, \quad -\infty < x < \infty, \quad t > 0,$$

subject to the arbitrary initial condition $u(x, 0) = f(x)$, by means of Fourier transforms. (Hint: You may find it useful to use the method of characteristics.)

- (b) Under what conditions can the formal solution in part (a) be made rigorous. (Hint: consider the function $f(x)$, its derivative or its integral, or)