

**Preliminary Examination: Partial Differential Equations,
Spring 2009**

Name: _____

#	possible	score
1	25	
2	25	
3	25	
4	25	
5	25	
Total	100	

There are 5 problems. Each problem is worth 25 points. You are required to do 4 of them. Please indicate which 4 you choose—Note: Only 4 problems will be graded. A sheet of convenient formulae is provided.

1. Fourier Series and Convergence behavior (25 points.)

Let

$$f(x) = x^2, \quad -\pi \leq x \leq \pi. \tag{1}$$

- (a) Determine the Fourier series for $f(x)$ on $[-\pi, \pi]$.
- (b) Use the Fourier series from part (a) to determine convergent formulas for $\pi^2/12$ and $\pi^2/6$.
- (c) State the theorem (s) that you used to guarantee convergence in part (b).
- (d) State the Riemann-Lebesgue Lemma, including any conditions on a function required for the Lemma to hold.
- (e) Explain why the Fourier Series in part (a) can be differentiated termwise and use the result to find a series converging to $\pi/6$.

2. Quasi-Linear first order PDE and Method of Characteristics (25 points.)

- (a) Use the method of characteristics to solve the following differential equation by setting up and solving at least 2 ODES

$$u_t + uu_x = 0, \tag{2}$$

with initial data

$$u(x, 0) = \begin{cases} 0, & x < 0 \\ x/\epsilon & 0 < x < \epsilon, \epsilon > 0 \\ 1, & x > \epsilon \end{cases} \tag{3}$$

- (b) Sketch the characteristics.
- (c) Now solve equation (2) with the following initial value problem:

$$u(x, 0) = \begin{cases} 0, & x < 0 \\ 1, & x > 0 \end{cases} \tag{4}$$

- (d) Extra Credit: Determine $u(x, t)$ for part (c). (Hint assume that $u(x, t) = g(x/t)$.)

3. Laplace's Equation in Polar Coordinates (25 points)

The Laplacian in Polar Coordinates is:

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \left(\frac{\partial^2 u}{\partial \theta^2} \right) = 0. \quad (3)$$

Further, assume that the domain of interest is defined by $0 < r \leq 1$, $0 \leq \theta < \alpha$, and $0 < \alpha < \pi/2$.

On this domain we impose Dirichlet boundary conditions $u(r, 0) = 0$, $u(r, \alpha) = 0$ and $u(1, \theta) = g(\theta)$. Finally, assume that special solutions to this equation are radially symmetric and have the form $u(r, \theta) = R(r)\Theta(\theta)$ (the solution of the problem is a sum of such separated solutions).

(a) Derive two ordinary differential equations for R and Θ under the assumption that the separation constant is μ .

(b) This is a Sturm-Liouville Problem on a compact domain in \mathbf{R}^2 , determine the form of the solution of the given PDE, $u(r, \theta)$. Determine formulas for the coefficients of the solution.

(c) State one theorem involving $g(\theta)$ that guarantees that the resulting generalized series will be convergent in each of the following ways: in L^2 , pointwise, and uniformly.

(d) State the Max-Min Theorem for this problem, and give an example of a non-constant function that satisfies this theorem.

4. Laplace Equation (25 points.) Consider the following equation

$$\nabla^2 u = f(x), \quad x \in D$$

$f(x) \in \mathbf{R}$, D is a simply connected domain in \mathbf{R}^n and

$$\alpha u + \beta \frac{\partial u}{\partial n} = \gamma, \quad x \in \partial D$$

(a) Assuming the existence of solutions to the above problem, establish uniqueness properties of the solution whenever possible; explain all steps.

(b) Take $n = 3$ dimensions. Find the decaying solution in terms of quadratures to

$$\nabla^2 u = f(|x|), \quad |x| < \infty$$

where $f(x) \in \mathbf{R}$, f integrable and $f(x) \rightarrow 0$ rapidly as $|x| \rightarrow \infty$.

(c) Find the leading asymptotic behavior of u as $|x| \rightarrow \infty$.

(d) State the Maximum principle.

5. Generalized Heat Equation (25 points)

Consider the following equation for $T = T(x, t)$

$$T_t = (k(x)T_x)_x + F(x)$$

on the interval $0 < x < L$ and where $t > 0$ and $F(x), k(x) > 0$ and $T(x, 0) = f(x)$.

For the boundary conditions below, find the solution $T(x, t)$ in terms of suitable functions; stipulate any conditions on $F(x), k(x), f(x)$ that are needed.

(a) $T(0, t) = T(t, L) = 0$.

(b) $T(0, t) = a(t), T(t, L) = b(t)$; stipulate any conditions on $a(t)$ and $b(t)$ that are needed.