

**PDE Preliminary Examination: Fall 2009**

Name: \_\_\_\_\_

There are 5 problems. Each problem is worth 25 points. You are required to do 4 of them. Please indicate which 4 you choose—Note: Only 4 problems will be graded. A sheet of convenient formulae is provided.

#	possible	score
1	25	
2	25	
3	25	
4	25	
5	25	
Total	100	

**1. Fourier Series and Convergence behavior.** Let

$$f(x) = |x| + x, \quad -\pi \leq x \leq \pi. \tag{1}$$

- (a) (5 points) Determine a Fourier series for  $f(x)$  on  $[-\pi, \pi]$ .
- (b) (5 points) What can you say about the convergence of this series? (E.g.: Mean convergence, pointwise convergence, uniform convergence, absolute convergence of Fourier coefficients, etc. ) Explain.
- (c) (5 points) Sketch the function (on  $\mathbb{R}$ ) to which your series converges.
- (d) (5 points) State the Riemann-Lebesgue Lemma, including any conditions on a function required for the Lemma to hold.
- (e) (5 points) Can this series be integrated term by term? To what function does the new series converge? Justify your answer.

**2. Quasi-Linear first order PDE and Method of Characteristics.**

(a) (15 points) Use the method of characteristics to solve the following differential equation by setting up and solving the characteristic ODES.

$$u_t + x^2 u_x = u^2, \tag{2}$$

with initial data  $u(x, 0) = e^{-x^2}$  for  $-\infty < x < \infty$ .

- (b) (5 points) Sketch the characteristics.
- (c) (5 points) On what domain in  $(x, t) \in \mathbb{R}^2$  does  $u$  exist?

**3. Laplace's Equation in Polar Coordinates.** Consider Laplace's equation

$$\Delta u = 0, \quad x \in D \tag{3}$$

where  $D \subset \mathbb{R}^2$  is the compact domain

$$D = \{(r, \theta) : 0 \leq r \leq 1, 0 \leq \theta \leq \alpha\}$$

for some  $0 < \alpha < \pi/2$ . On this domain we impose Dirichlet boundary conditions

$$u(r, 0) = 0, \quad u(r, \alpha) = 0, \quad u(1, \theta) = g(\theta).$$

(a) (10 points) Determine a formal solution of (3) using separation of variables. Recall that the Laplacian in polar coordinates is

$$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2}.$$

(b) (5 points) State conditions on  $g(\theta)$  that guarantee that the resulting generalized series will be convergent in each of the following ways: in  $L_2$ , pointwise, and uniformly.

(c) (10 points) State Maximum and Minimum principles for this problem and give an example of a non-constant function that illustrates these results.

**4. Generalized Heat Equation.** The temperature  $T(x, t)$  in a long metal bar with nonuniform thermal conductivity obeys the equation

$$\frac{\partial T}{\partial t} = \frac{\partial}{\partial x} \left( k(x) \frac{\partial T}{\partial x} \right) + F(x), \quad 0 < x < L, \quad t > 0 \quad (4)$$

where  $F(x) > 0, k(x) > 0$  and  $T(x, 0) = f(x)$ .

For the boundary conditions below, find the solution  $T(x, t)$  in terms of suitable functions; stipulate any conditions on  $F(x), k(x)$ , and  $f(x)$  that are needed.

(a) (10 points)  $T(0, t) = T(x, L) = 0$ .

(b) (10 points)  $T(0, t) = a(t), T(x, L) = b(t)$ ; stipulate any conditions on  $a(t)$  and  $b(t)$  that are needed.

**5. Eigenvalues and Eigenfunctions of the Laplacian** Let  $D$  be a simply connected bounded domain in  $\mathbb{R}^2$ . Consider the Helmholtz equation

$$\Delta u + \lambda u = 0, \quad (5)$$

with Dirichlet boundary conditions. Assume  $\lambda \in \mathbb{C}$  is an eigenvalue and is nontrivial (*i.e.*, non-zero). Assume that exists a function  $u$  that solves this problem; this  $u$  is an eigenfunction. (This is all in the context of a suitable inner product.) In fact, we assume that there are *infinitely many* eigenvalues  $\lambda_n$  and eigenfunctions  $u_n, n = 0, \dots, \infty$ , and moreover that all eigenvalues are simple (*i.e.*, their eigenspaces are one-dimensional).

(a) (5 points) Show that all eigenvalues are real and positive.

(b) (5 points) Show that eigenfunctions corresponding to different eigenvalues are orthogonal.

(c) (10 points) Consider Laplace's equation  $\Delta u = 0$  on the cylinder  $D \times [0, H] \subset \mathbb{R}^3$  (not necessarily circular) with base  $D \subset \mathbb{R}^2$  and height  $H$ . Assume that

$$\begin{aligned} u(x, y, 0) &= 0, & (x, y) &\in D \\ u(x, y, z) &= 0, & (x, y, z) &\in \partial D \times [0, H] \\ u(x, y, H) &= 1, & (x, y) &\in D. \end{aligned}$$

Find a formal solution to this boundary value problem in terms of the eigenfunctions of the Helmholtz equation (5).