Department of Applied Mathematics
PROBABILITY AND STATISTICS PRELIMINARY EXAMINATION
August 20, 2003

Notice: Do four of the following five problems. Place an X on the line opposite the number of the problem that you are **NOT** submitting for grading. Please do not write your name anywhere on this exam. You will be identified only by your student number, given below and on each page submitted for grading. Show all relevant work.

1. ____
2. ____
3. ____
4. ____
5. ____

Total ____

Student Number ____________________

1. A large number N, of people are subject to a blood test. This can be administered in two ways. (i) Each person can be tested separately. In this case N tests are required. (ii) The blood samples of k ($k \leq N$) people can be pooled and analyzed together. If the test is *negative*, this one test suffices for the k people. If the test is *positive*, each of the k persons must be tested individually. Assume that the probability p that an individual test is positive is the same for all people and that all people are independent.

   (a) What is the probability that the test for a pooled sample of k people is positive?

   (b) What is the expected number of tests necessary under plan (ii)?

   (c) Find a simple equation which will approximately minimize (in k) the expected number of tests in (b). "Approximately" means under the assumption that both p and kp are "small".

2. A line of length L is divided in half. The point X is chosen from $(0, L/2)$ according to the density function $f_X(x) = 8x/L^2$ for $0 \leq x \leq L/2$. The point Y is chosen uniformly from $(L/2, L)$. Find the probability that the distance between the two points is greater than $L/3$.

3. Suppose that users arrive at a public library according to a Poisson process $\{N(t)\}_{t\geq 0}$ with parameter λ. The amount of time spent in the library by any user is a random variable with cumulative distribution function $B(\cdot)$ and the times spent in the library by the different users are mutually independent and independent of the arrival times $\{N(t)\}_{t\geq 0}$. Suppose that the library opens at time 0 and let $N_1(t)$ be the number of users in the library at time $t > 0$.

   (a) What is the distribution of $N_1(t)$?

   (b) Give an explicit computation of $\mathrm{E}[N_1(t)]$ when $B(x) = 1 - \exp(-\mu x)$.

   (c) Derive an expression for $\lim_{t\to\infty} \mathrm{E}[N_1(t)]$.

4. Let $X_i = \sqrt{Y_i}\, Z_i$ with $Z_i \sim \mathrm{N}(0,1)$ and Y_i such that its Laplace transform is given by

$$L_Y(t) = \mathrm{E}[\exp(-tY_i)] = \exp(-ct^{\alpha/2})$$

for some constant $c > 0$ and $0 < \alpha < 2$. We assume that the $\{Y_i\}_{i=1,\ldots,n}$ and the $\{Z_i\}_{i=1,\ldots,n}$ are independent samples.

(a) Compute the characteristic function of X_j, that is, compute $\mathrm{E}[\exp(itX_j)]$.

(b) Does the second moment of X_j, $E(X_j^2)$ exist for all $0 < \alpha < 2$? If it does not exist, give a counter-example..

(c) Let $X_i(y) = [X_i | Y_i = y]$. Compute the limiting distribution (as $n \to \infty$) of

$$\bar{X}_n(y) = \frac{1}{n}\sum_{i=1}^{n} X_i(y) \qquad \text{and of} \quad n^{(1-1/\alpha)}\bar{X}_n(y).$$

(d) What is the limiting distribution of

$$X_{\alpha,n} = \frac{1}{n^{(1/\alpha)}}(X_1 + \cdots + X_n) = \frac{n}{n^{(1/\alpha)}}\bar{X}_n = n^{(1-1/\alpha)}\bar{X}_n.$$

(e) Do the results of part (c) contradict the results of part (d)?

5. Let $X_1, X_2, \ldots, X_n$ be a random sample from the Poisson distribution with rate λ. Define $\theta := P(X = 0)$.

(a) Find the UMVUE (unique minimum-variance unbiased estimator) for θ. (Hint: Consider functions of the form $\hat{\theta} = c^{\sum X_s}$ for some c.)

(b) Find the Cramer-Rao lower bound (CRLB) for the variance of all unbiased estimators of θ.

(c) Does the variance of your UMVUE attain the CRLB?

(d) Define $Y = \frac{1}{n}\sum_{i=1}^{n} I_{X_n=0}$, where $I_.$ is the indicator function. Observe that Y is also an unbiased estimator of $\theta = e^{-\lambda}$. Find the variance of Y.