Department of Applied Mathematics
PROBABILITY AND STATISTICS PRELIMINARY EXAMINATION
August 17, 2005

<u>Notice</u>: Do four of the following five problems. Place an X on the line opposite the number of the problem that you are **NOT** submitting for grading. Please do not write your name anywhere on this exam. You will be identified only by your student number, given below and on each page submitted for grading. Show all relevant work.

1. ____
2. ____
3. ____
4. ____
5. ____

Total ____

Student Number ____________________________

1. Suppose that the probabilty density function of a random variable X is given by

$$f_X(x) = \frac{1}{2}x^2e^{-x}, \qquad \text{if } x > 0$$

and is zero elsewhere. Suppose that given $X = x$ ($x > 0$), the random variables Y_1 and Y_2 are independent and identically distributed with

$$Y_i \sim uniform(0, x).$$

Find the conditional expectation

$$\mathsf{E}[X|Y_1 = y_1, Y_2 = y_2].$$

2. A random variable Y with probability density function (pdf) $g(x)$ is called *stochastically bigger* than a random variable X with pdf $f(x)$ if $P(Y > u) \geq P(X \geq u)$ for all $u \in \Re$. Prove that if the likelihood ratio $g(x)/f(x)$ is monotone increasing, then Y is stochastically bigger than X. For simplicity, you may assume that f and g are continuous.

3. Suppose that male customers arrive in a store according to a Poisson process with rate λ and that female customers arrive according to a Poisson process with rate μ. Furthermore, suppose that these two processes are independent.

   (a) Suppose during the first hour that one male and one female customer arrived. What is the probability that the male customer arrived before the female?

   (b) What is the expected number of males to arrive between two successive female arrivals?

4. Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from the distribution with probability density function

$$f(x;\beta) = 3\beta x^2 e^{-\beta x^3} \qquad \text{for } x > 0.$$

   (a) Find the <u>form</u> of the critical region for the most powerful test of size α for testing $H_0 : \beta = \beta_0$ versus $H_1 : \beta = \beta_1$, where $\beta_1 < \beta_0$. (An example of the phrasing of an answer would be: "reject H_0 if $\sum X_i > c$ where c is chosen such that...")

   (b) If you haven't done so already, show that your test from part (a) can be written in terms of a chi-squared critical value.

   (c) Is your test above uniformly most powerful for $H_0 : \beta = \beta_0$ versus $H_1 : \beta < \beta_0$? Justify/explain your answer.

   (d) Does there exisit a uniformly most powerful test for $H_0 : \beta = \beta_0$ versus $H_1 : \beta \neq \beta_0$? Justify/explain your answer.

5. A ski-lift cabin in Steamboat Springs takes a group of passengers to the top of a mountain and back down. Every 10 minutes (at 0, 10, 20, ... minutes past the hour), a cabin leaves in each direction, and the journey up or down the mountain lasts 4 minutes.

   Alice and Ralph attempt to board a cabin going up the mountain just as it is about to leave. Alice manages to get in while Ralph is left behind. The pair try to get together again, but do not know each other's movements. Until they meet, they both travel between the locations "T" (top of the mountain) and "B" (bottom of the mountain) according to independent Markov chains with state space $\{\text{T}, \text{B}\}$ and transition matrices

$$\mathbf{P}_{Alice} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} \qquad \text{and} \qquad \mathbf{P}_{Ralph} = \begin{bmatrix} 2/3 & 1/3 \\ 1 & 0 \end{bmatrix},$$

   respectively, where each transition corresponds to a cabin traveling in each direction (i.e. to a time interval of 10 minutes). Assume that Alice and Ralph find each other as soon as they are in a single location.

   (a) Let X_n be the ordered pair of the locations of Alice and Ralph, respectively, immediately after the arrival of the nth cabin in each direction. For example, we know that $X_0 = (T, B)$. Give the transition probability matrix of the Markov chain $\{X_n\}$.

   (b) Calculate the expected time (in minutes) until Alice and Ralph find each other. (Time is measured from the moment that Alice arrives at the top of the mountain for the first time.)

   (c) Calculate the variance of the time until Alice and Ralph find each other.