Syllabus for Applied Analysis Preliminary Exam

The principal reference for this syllabus is:

[1] J.K. Hunter and B. Nachtergaele, *Applied Analysis.*

With the exception of the topics from advanced calculus, a question may appear on the exam if and only if the topic is covered in one of the sections of [1] listed below. Recommended supplemental references include:

[2] A. Friedman, *Foundations of Modern Analysis.*
[3] W. Fulks, *Advanced Calculus.*
[4] P. Lax, *Functional Analysis.*
[5] A.N. Kolmogorov and S.V. Fomin, *Introductory Real Analysis.*
[6] H. Royden, *Real Analysis.*

Advanced Calculus:

- Integration of functions of several variables: line and volume integrals in 2D, line, surface, and volume integrals in 3D.
- Differentiation: gradient, curl, divergence, Jacobian. Connection between rotation-free vector fields and potential fields.
- Partial integration, Green's theorems, Stokes' theorem, Gauss' theorem. The consequences of these theorems for vector fields that are divergence or rotation free.
- The concepts max, min, sup, inf, $\limsup$, $\liminf$, lim.
- Convergence criteria for sequences and series.

Metric and Normed Spaces:

- The topology of metric spaces. Normed spaces. Cauchy sequences. Compactness. Completeness. *Sections 1.1 – 1.7.*
- Basic properties of the space of continuous functions on a metric space. The Arzelà-Ascoli theorem. *Sections 2.1, 2.2, and 2.4.*
- The contraction mapping theorem. *Sections 3.1 and 3.3.*

Banach Spaces:

- Bounded linear maps between Banach spaces, different topologies on the set of bounded operators between Banach spaces. The kernel and the range of a linear map. Connection between an operator being coercive and having closed range. The exponential of a bounded operator on a Banach space. *Sections 5.1 – 5.5.*
- The dual of a Banach space. The Hahn-Banach theorem. Compactness of the unit ball in the weak topology on reflexive spaces. *Section 5.6 (weak-* convergence and the full Banach-Alaoglu theorems are not included).*

Hilbert Spaces (separable spaces only):

- Orthogonal sets and orthonormal bases. Bessel's inequality. Parallelogram law and polarization identity. *Sections 6.1 – 6.3.*
- Riesz representation theorem. *Section 8.2.*
- Fourier series. Parseval identity. Convolutions. *Section 7.1.*
- Sobolov spaces on the d-dimensional torus. Sobolev embedding. *Section 7.2.*

Linear Operators on Hilbert Spaces:

- The adjoint of an operator. Self-adjoint, normal, and unitary operators. Projections. $\overline{\text{ran}A} = (\text{ker}A^*)^\perp$. Fredholm alternative. *Sections 8.1, 8.3, 8.4.*
- The spectrum of general operators on a Hilbert space. Basic properties of the resolvent operator. The spectral theorem for self-adjoint compact operators. Functions of operators. *Sections 9.2, 9.3, 9.4, 9.5.*

Measure Theory, Integration, and L^p-spaces:

- Basic properties of Lebesgue measure. Nullsets. "Almost everywhere" and "essential supremum". *Sections 12.1 and 12.2 (but only the concepts listed here).*
- Definition of the Lebesgue integral. *Section 12.3.*
- Convergence theorems: Fatou's lemma, Monotone convergence, Lebesgue dominated convergence. *Section 12.4.*
- Fubini's theorem (with respect to Lebesgue measure on $\mathbb{R}^d$ only). *Section 12.5.*
- Definition of $L^p(X, \mu)$ for a measure space (X, μ). Hölder's and Minkowski's inequalities. The dual of $L^p(X, \mu)$ for $p \in [1, \infty)$. Density of simple functions, and compactly supported smooth functions in $L^p(\mathbb{R}^d)$. *Sections 12.6, 12.8, and parts of 12.7.*