Building on the Legacy of John Greene:
The Transition to Chaos in Volume-Preserving Maps

J. D. Meiss
University of Colorado at Boulder

John M. Greene
1928-2007

"You are trying to solve the inverse scattering problem"
"Oh, that!"

1950 BS Cal Tech
1956 PhD Univ. of Rochester
1955-1982 PPPL
1982-1995 General Atomics

1992 APS Maxwell Prize
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MHD Instabilities
BGK Modes
Inverse Scattering
Greene's Residue Criterion
MHD Hamiltonian Theory
Magnetic Nulls

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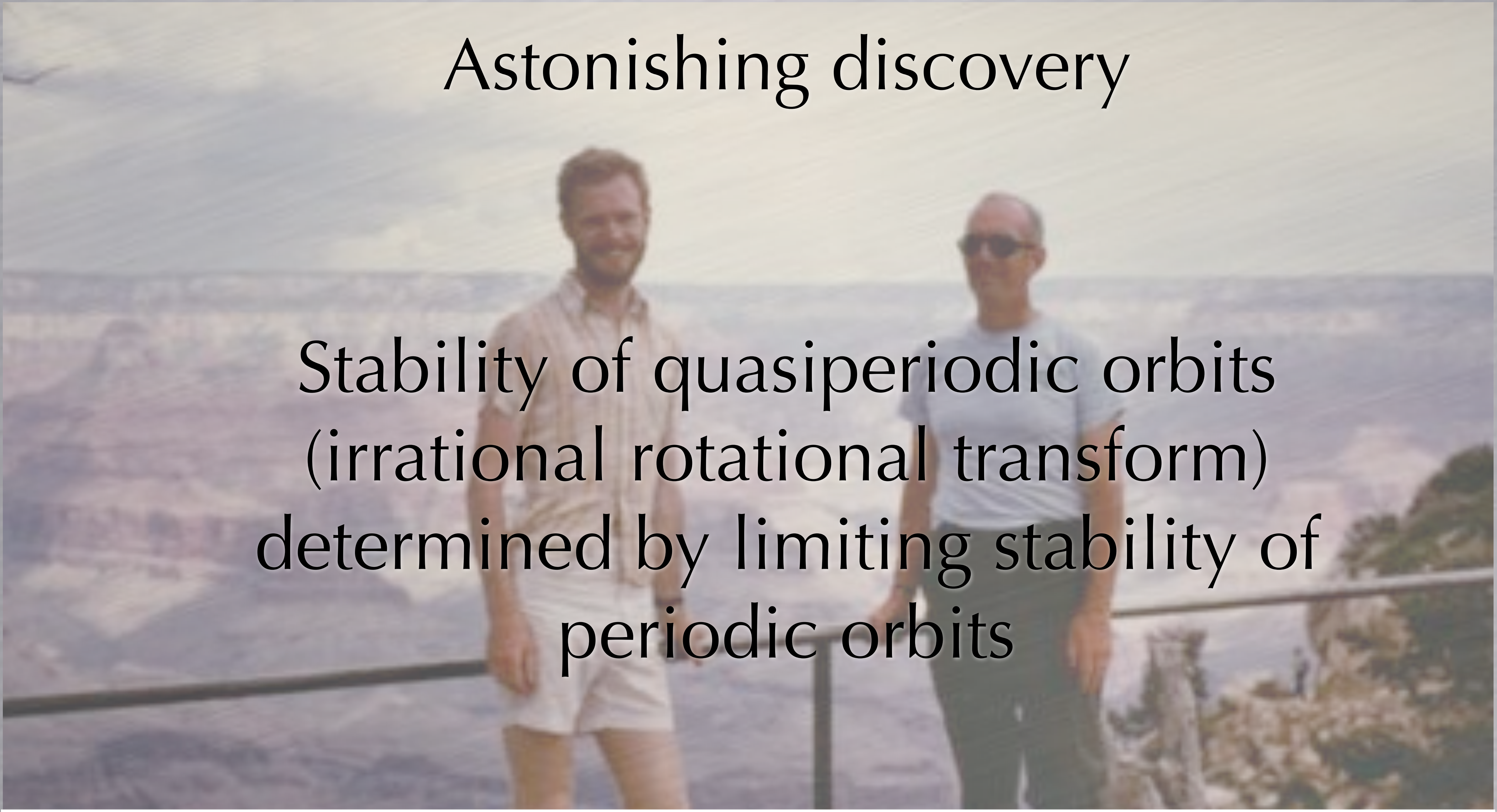
The Residue Criterion



Astonishing discovery

Stability of quasiperiodic orbits (irrational rotational transform) determined by limiting stability of periodic orbits

Greene, J. M. (1968). "Two-Dimensional Area Preserving Mappings." J. Math Physics **9**: 760-768.
Greene, J. M. (1979). "A Method for Computing the Stochastic Transition." J. Math. Physics **20**: 1183-1201.
Greene, J. M. (1980). The Calculation of KAM Surfaces. <u>Nonlinear Dynamics</u>. Ann. New York Acad.**357**: 80-89.

- Area Preserving Map

$$(x', y') = (x + y', y - \frac{k}{2\pi}\sin(2\pi x))$$

- Stability of Periodic Orbits

$$Df^n = \frac{\partial(x_n, y_n)}{\partial(x_0, y_0)} \quad \Rightarrow \lambda_1, \lambda_2$$

$$\det(\lambda I - Df^n) = \lambda^2 - \tau\lambda + 1\,, \quad \tau = Tr(Df^n)$$

- Residue

$$R = \frac{1}{4}(2 - Tr(Df^n))$$

R<0: Hyperbolic
0<R<1: Elliptic
R>1: Reflection Hyperbolic

- sequence of periodic orbits, rotation numbers $\frac{m_i}{n_i} \to \gamma$ Use Continued fractions...
- bounded Residue implies existence of invariant circle

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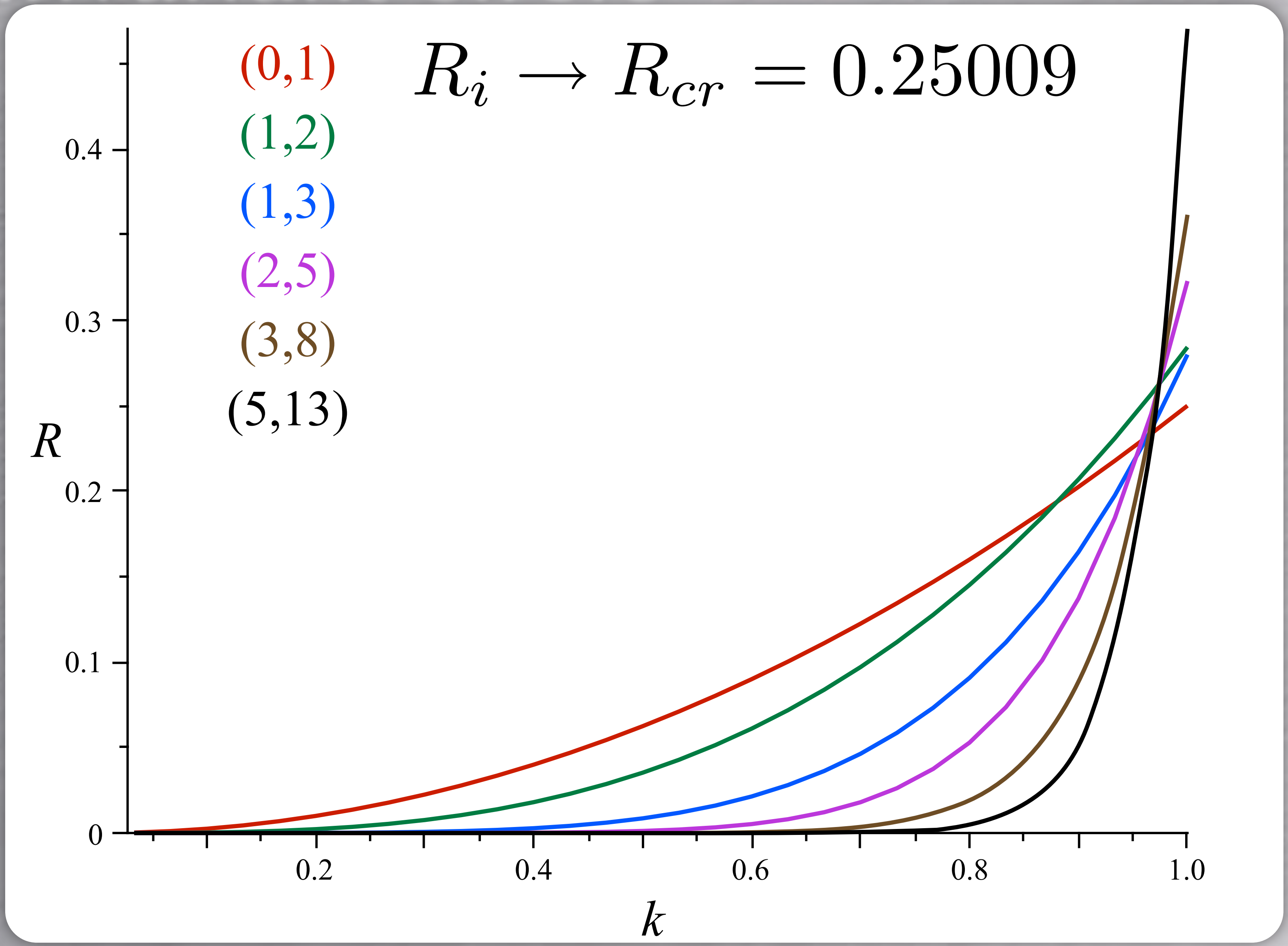

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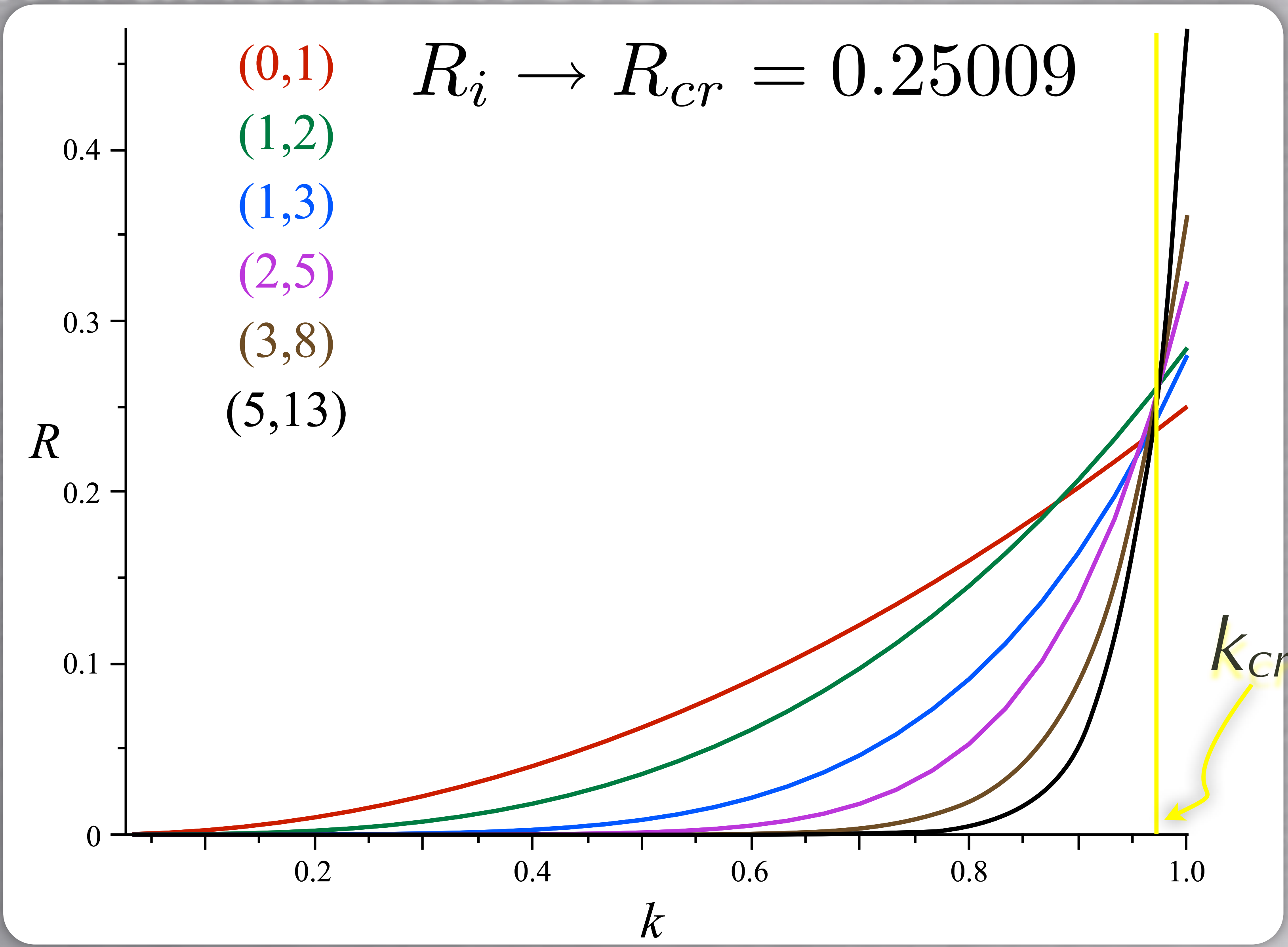

k_{cr}=0.97163540631

- sequence of periodic orbits, rotation numbers $\frac{m_i}{n_i} \to \gamma$

Use Continued fractions...

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- Last Invariant Circle:

  γ = Golden Mean!

- Much has been Proved:
  - MacKay, R. S. (1992). "On Greene's Residue Criterion." Nonlinearity 5(1): 161-187.
  - Delshams, A. and R. de la Llave (2000). "KAM Theory and a Partial Justification of Greene's Criterion for Nontwist Maps." SIAM J. Math. Anal. 31(6): 1235-1269.

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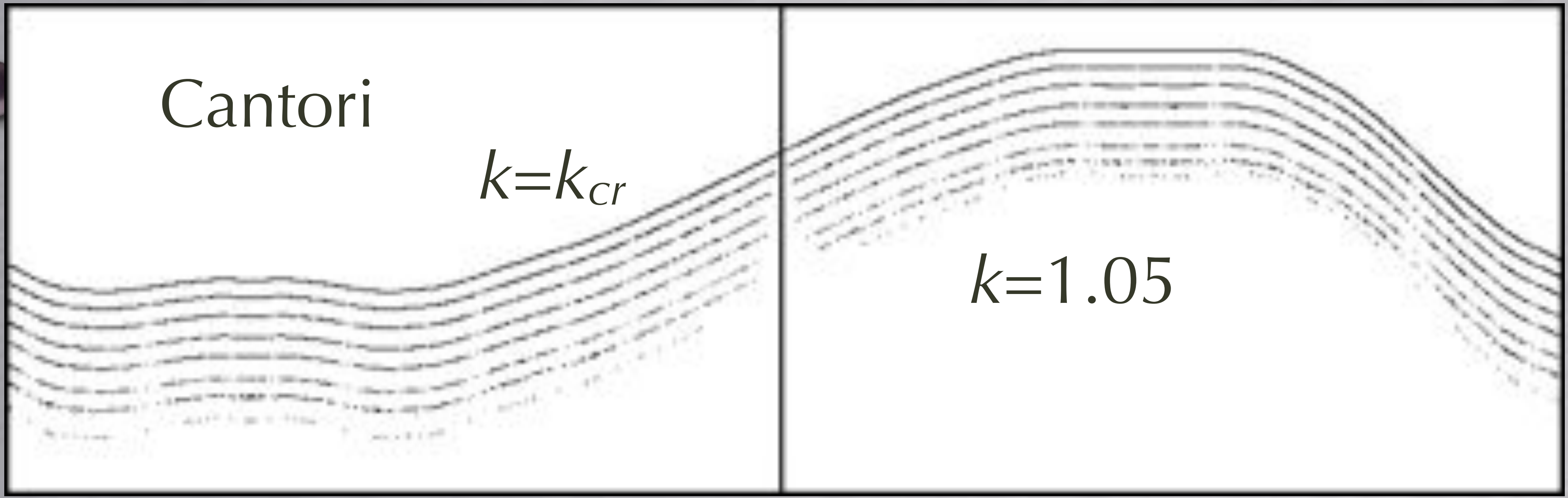

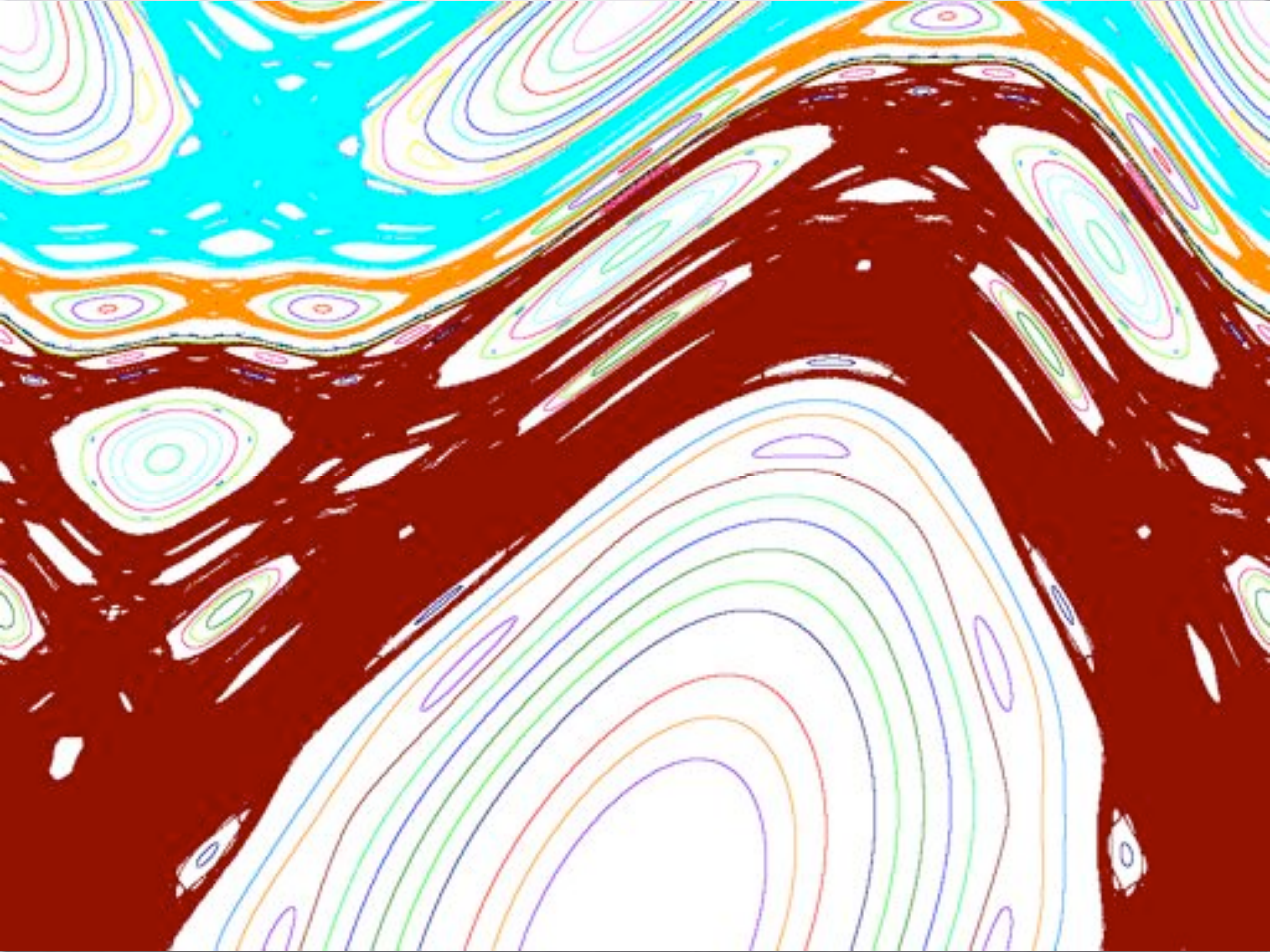

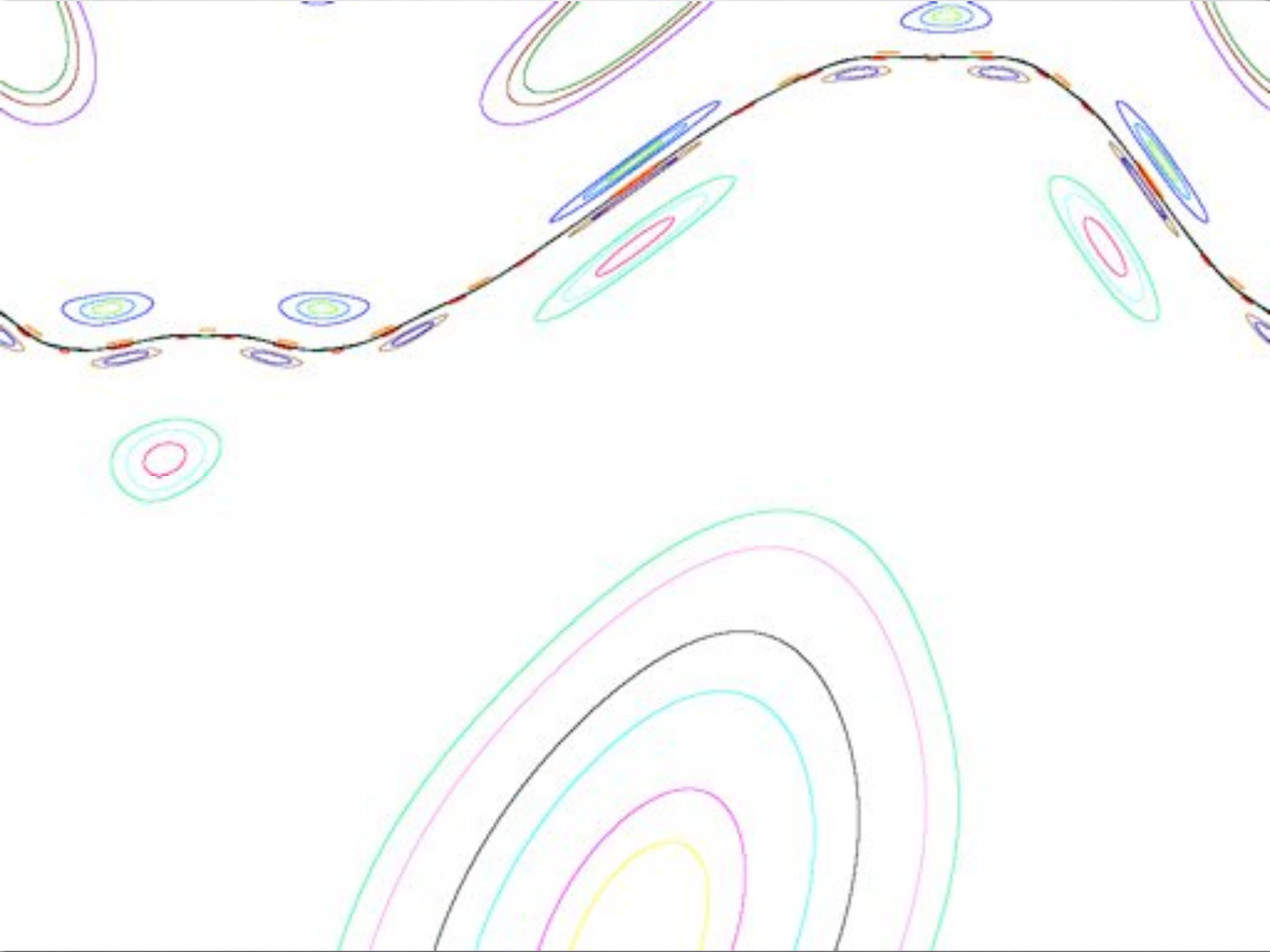

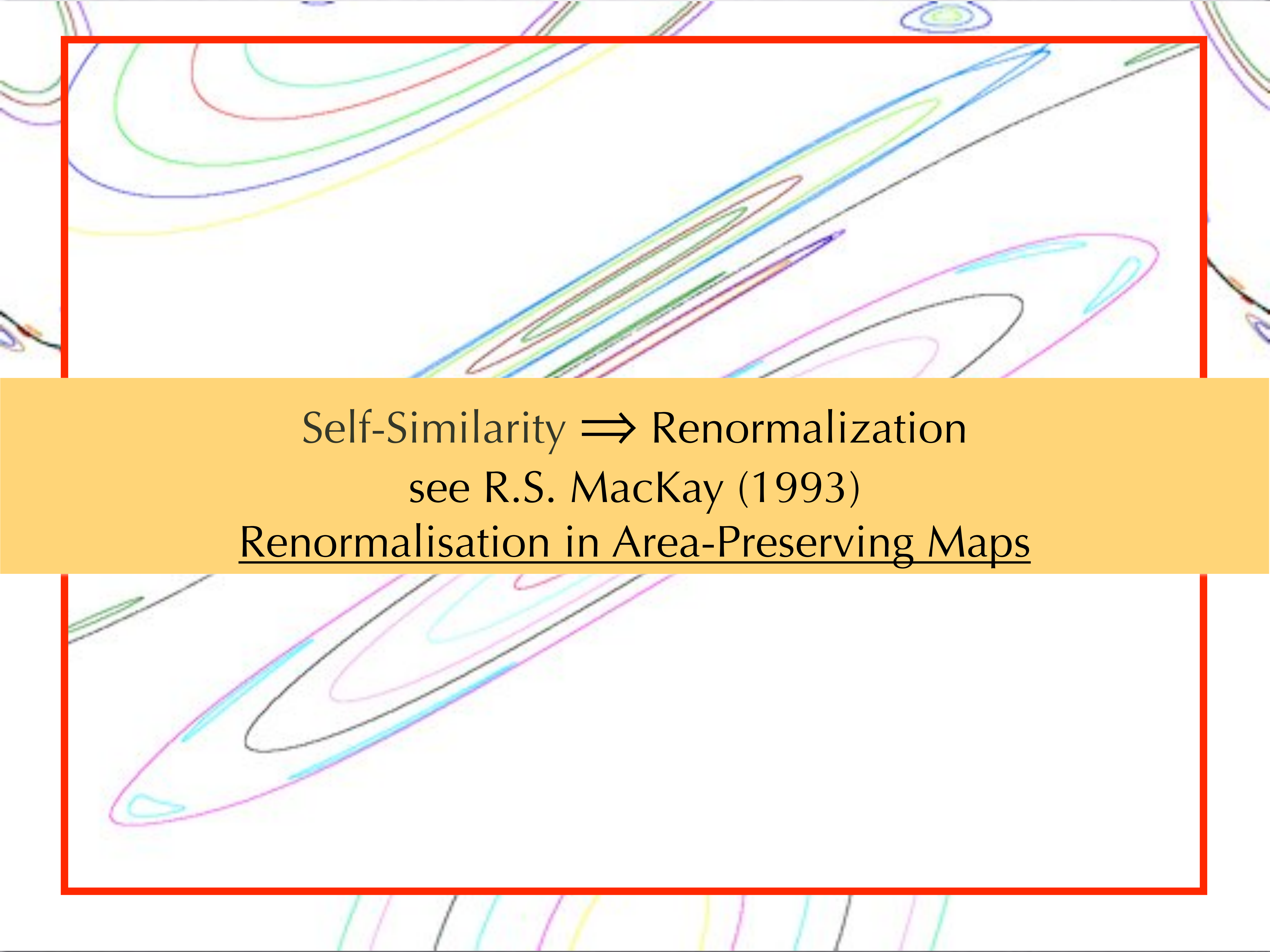

Self-Similarity $\Longrightarrow$ Renormalization

see R.S. MacKay (1993)

<u>Renormalisation in Area-Preserving Maps</u>

- Generalize to Symplectic Twist Maps?

$$\begin{aligned} \vec{x}\,' &= \vec{x} + \vec{y}\,' \\ \vec{y}\,' &= \vec{y} - \nabla V(\vec{x}) \end{aligned}$$

Froeshlé Map

- ∃ ($\vec{m}$,n) Periodic Orbits
  - Kook, H. T. and J. D. Meiss (1989). "Periodic-Orbits for Reversible, Symplectic Mappings." Physica D 35(1-2): 65-86.
- ∃ Tori and a last torus: "KAM" and "Converse KAM"
  - MacKay, R. S., J. D. Meiss and J. Stark (1989). "Converse KAM Theory for Symplectic Twist Maps." Nonlinearity 2: 555-570.
- ∃ Cantori: "Anti-Integrable Limit"
  - MacKay, R. S. and J. D. Meiss (1992). "Cantori for Symplectic Maps near the Anti-Integrable Limit." Nonlinearity 5: 149-160.
- ∃? Residue Criterion: Symplectic ⇒ Reflexive

$$\lambda^4 - \tau\lambda^3 + \sigma\lambda^2 - \tau\lambda + 1 \qquad R_{1,2} = \frac{1}{4}\left(2 - \lambda_i - \frac{1}{\lambda_i}\right)$$

  - Tompaidis, S. (1999). Approximation of Invariant Surfaces by Periodic Orbits in High-Dimensional Maps. <u>Hamiltonian Systems with Three or More Degrees of Freedom (S'agaro, 1995)</u>.

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But what is the generalization of the *golden mean?*
(No satisfactory generalization of continued fractions)

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$\lambda^4 -$ … $\frac{1}{\lambda_i}\Big)$

  - Tompai … Dimensional Maps. H…

But are there critical residues or multiple pathways to destruction?

Volume Preserving Maps

Magnetic Field line flows

$$\frac{dx}{dt} = B(x,t) \quad \nabla \cdot B = 0$$

Incompressible Fluids

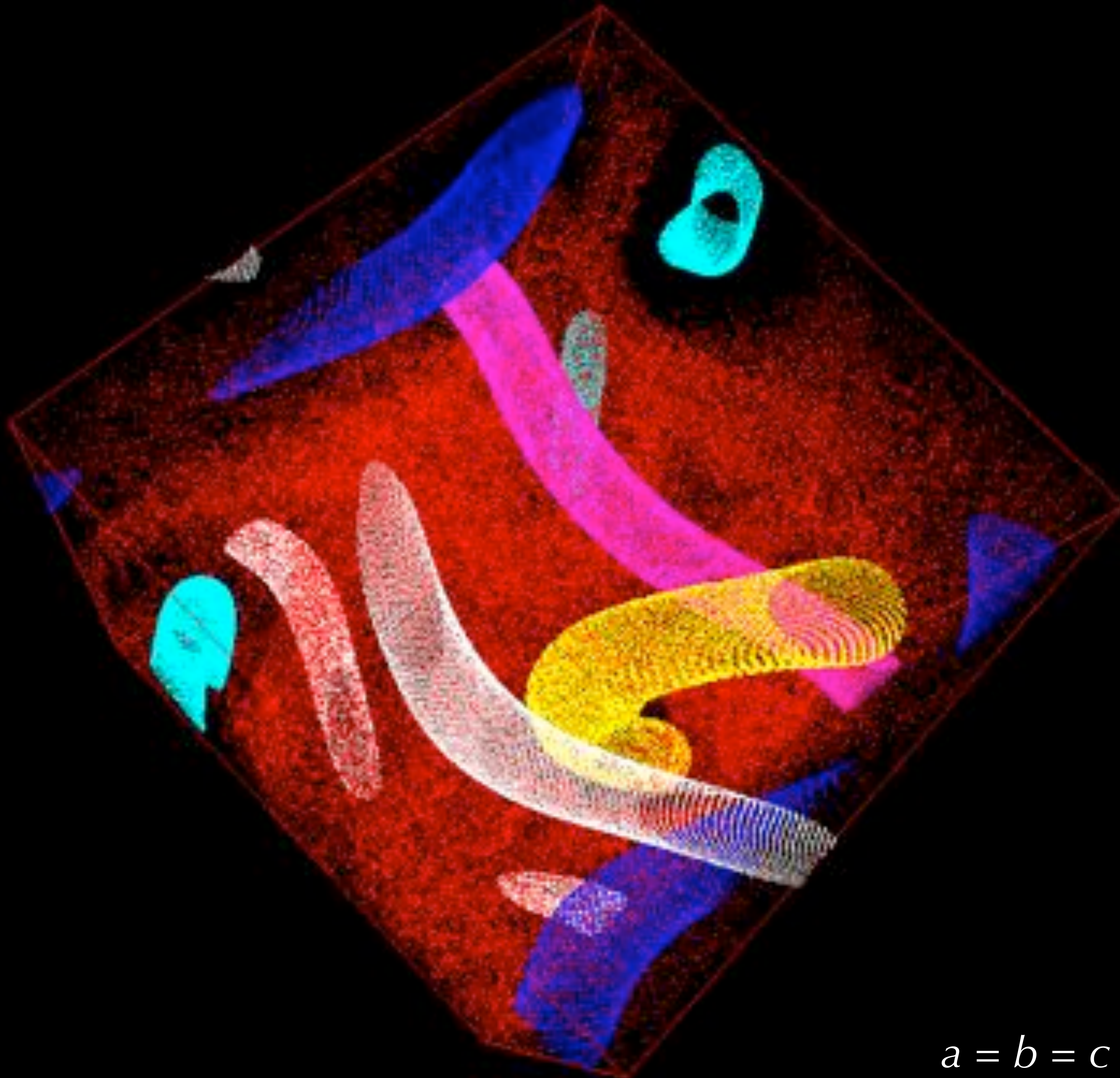
$$\frac{dx}{dt} = v(x,t) \quad \nabla \cdot v = 0$$

Poincaré Map for Periodic Time dependence: V.P.

ABC Map

$$
\begin{aligned}
x' &= x + a\sin(2\pi z) + c\cos(2\pi y) \\
y' &= y + b\sin(2\pi x') + a\cos(2\pi z) \\
z' &= z + c\sin(2\pi y') + b\cos(2\pi x')
\end{aligned}
$$

$a = b = c = 0.1$



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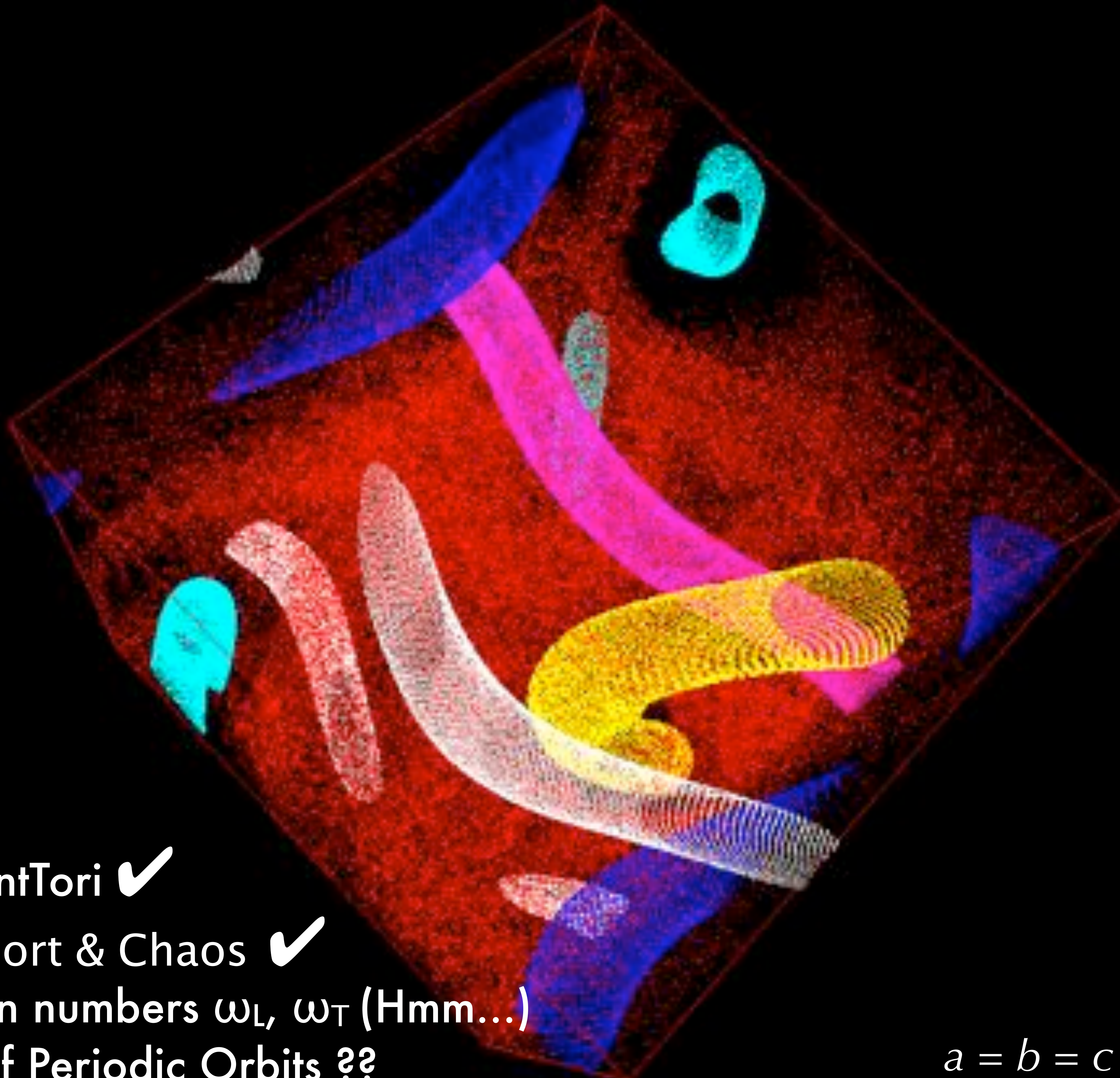


InvariantTori ✔

Transport & Chaos ✔

Rotation numbers ω_L, ω_T (Hmm...)

Limits of Periodic Orbits ??

$a = b = c = 0.1$

A Residue Criterion?

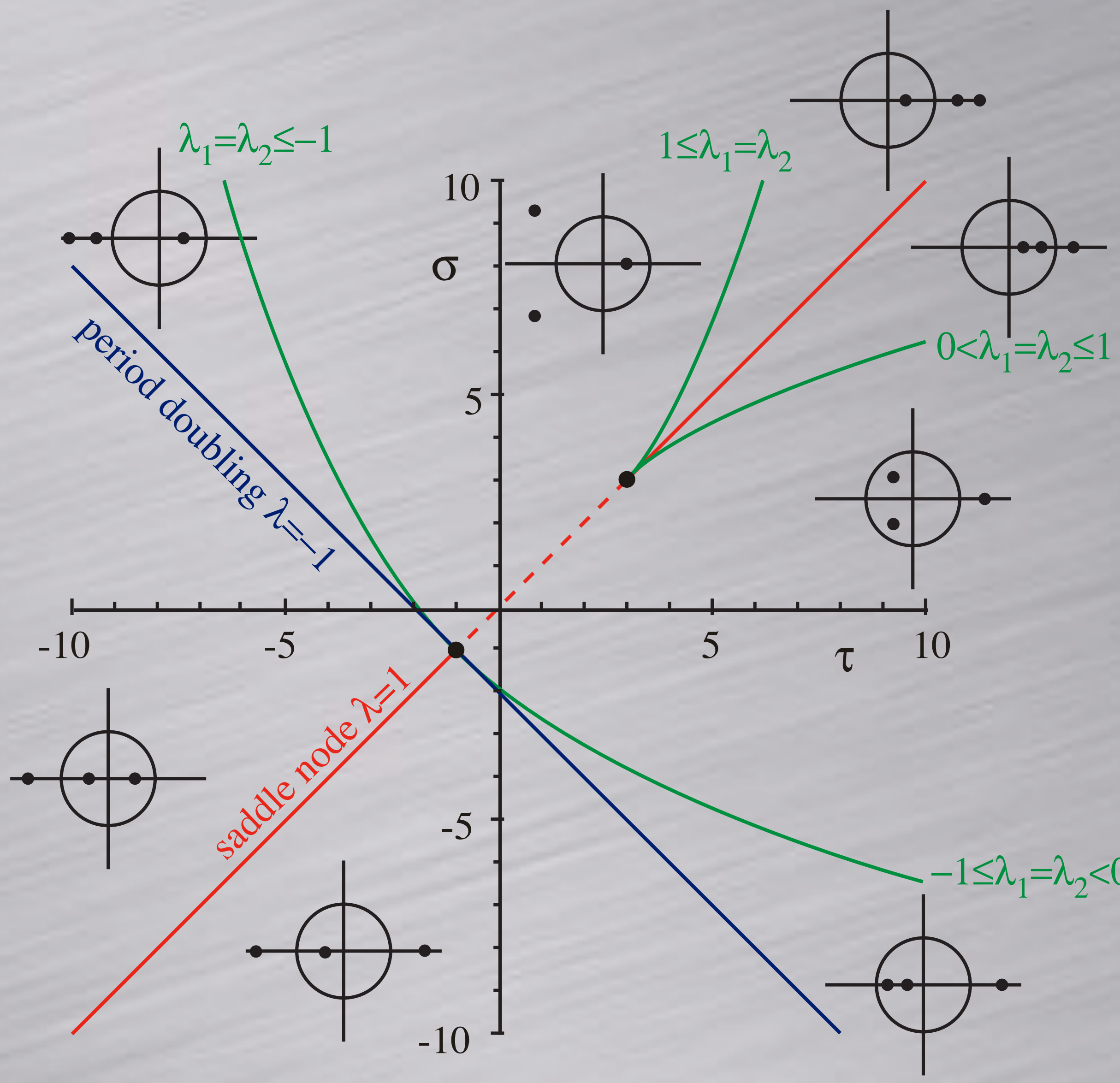
- KAM theory applies (Cheng & Sun)
  - However, can't fix the frequencies!
- Is there a last torus? Self-Similarity?
  - What rotation vector plays the role of the golden mean?
  - Perhaps the spiral mean $\sigma^3 = \sigma+1$?
- Are there cantori?
  - Anti-integrable theory by Li & Malkin

Stability

- Characteristic Polynomial has two parameters $\lambda^3 - \tau\lambda^2 + \sigma\lambda - 1 = 0$

$$\tau = Tr(Df)$$

$$\sigma = \frac{1}{2}(\tau^2 - Tr(Df^2))$$



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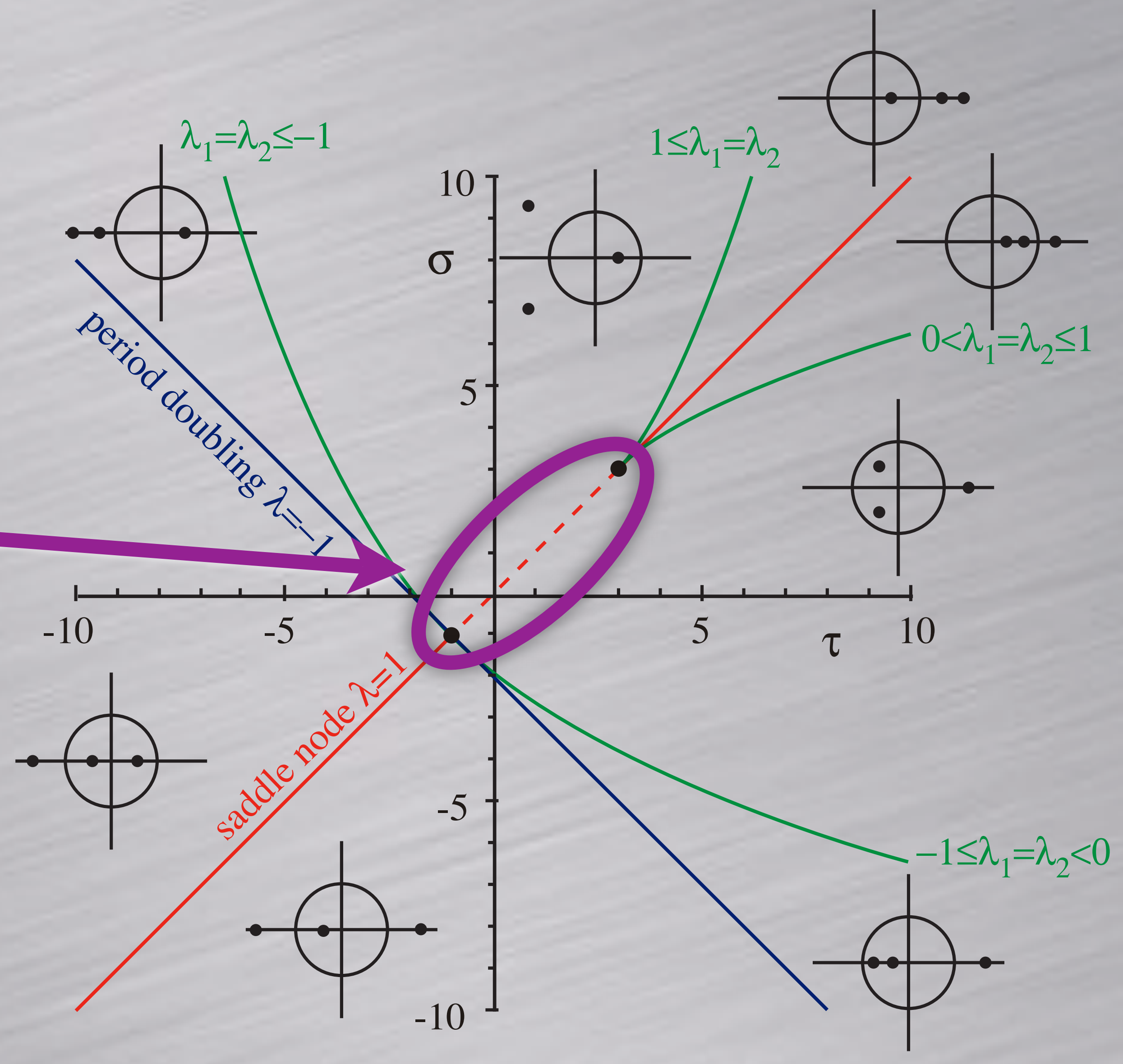
Stable Tori would be here:
Saddle-Center-Hopf Line



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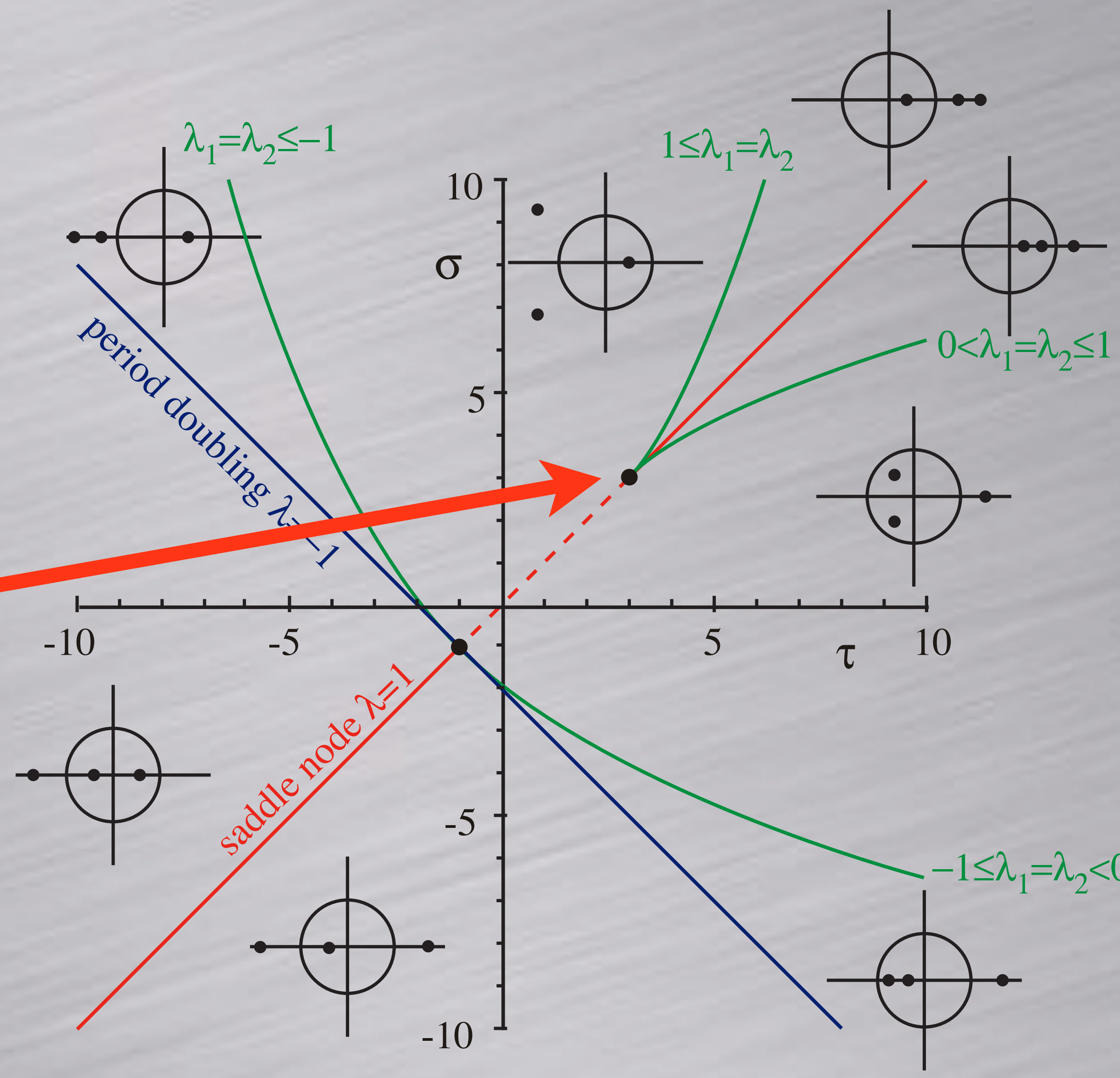
Triple-one Eigenvalue



Quadratic VP Map

- Normal form for (1,1,1) bifurcation
- Generalizes Hénon's 2D Map

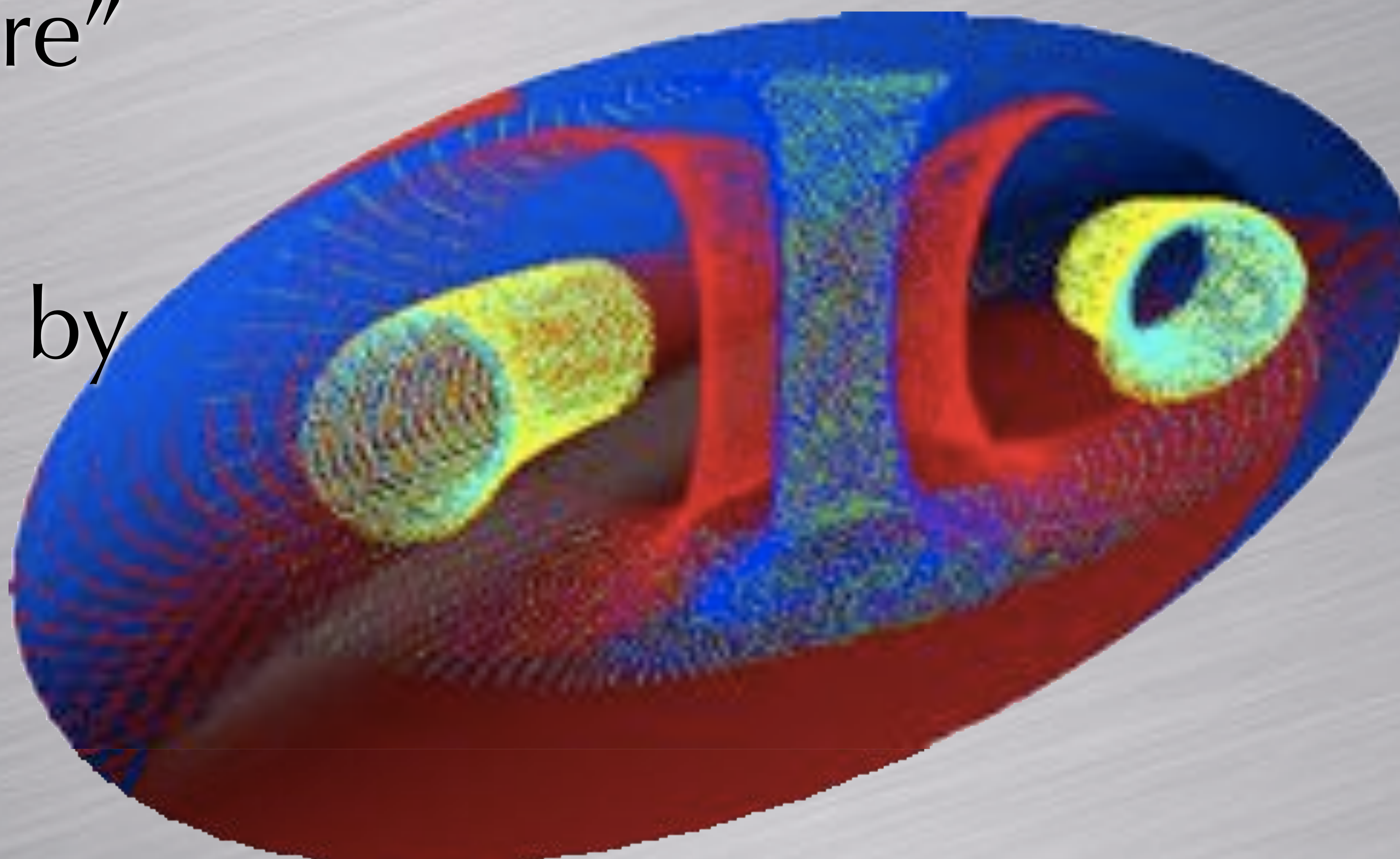
$$f(x,y,z) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z + p(x,y) \end{pmatrix}$$

$$p(x,y) = -\varepsilon + \mu y + ax^2 + bxy + cy^2$$

- Hydrogen-Atom Model for 3D dynamics

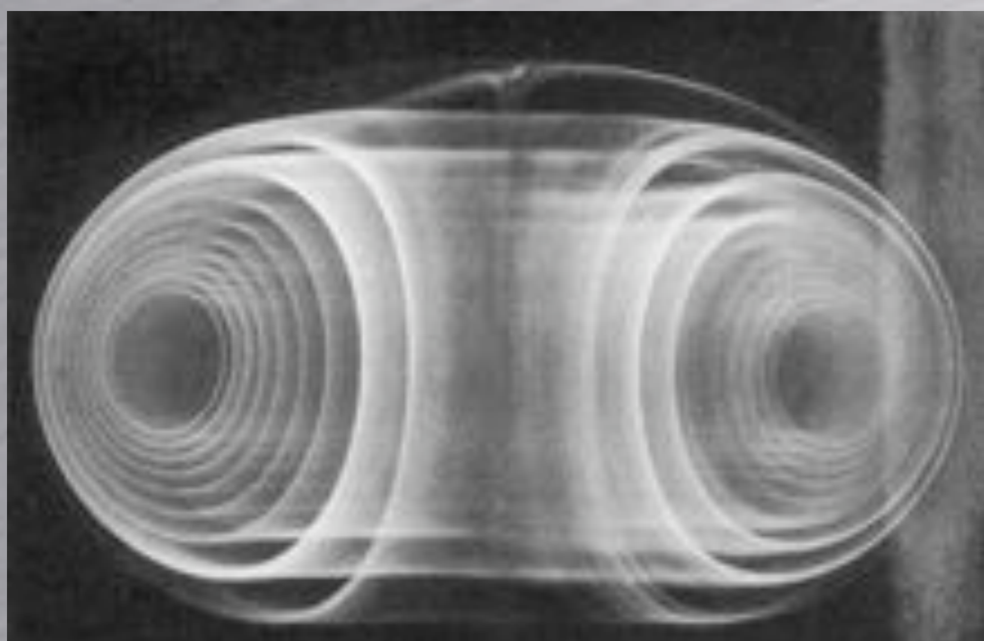
Saddle-Center-Hopf

- Two fixed points: typically one *type-A* (2,1) and one *type-B* (1,2). (e.g. Greene's Magnetic Nulls)
- 2D stable and unstable manifolds intersect forming a "sphere"
- Inside of sphere foliated ($\varepsilon << 1$) by invariant tori
- Spheromak is generic!

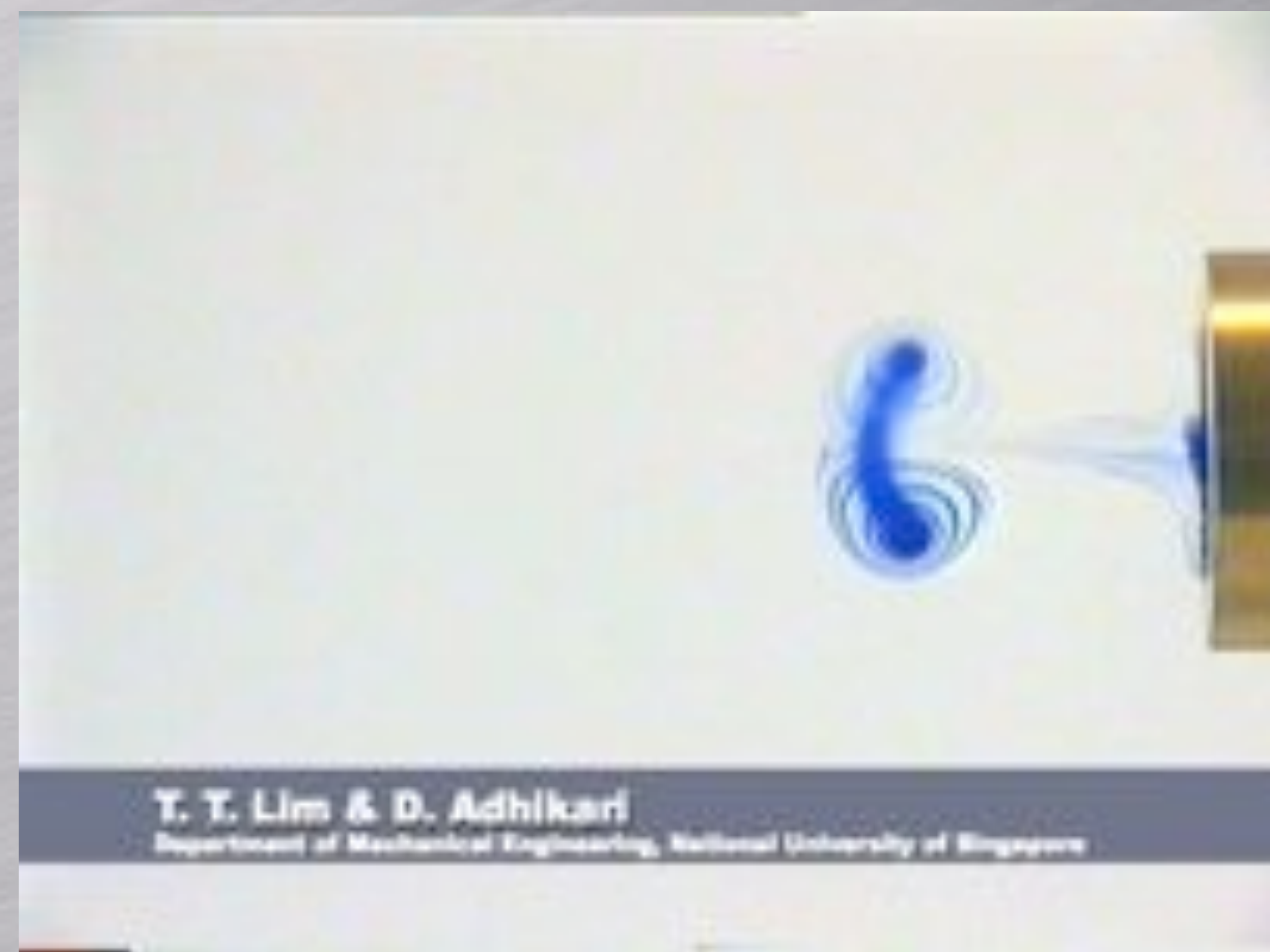


Dullin & Meiss 2008

Vortex Rings



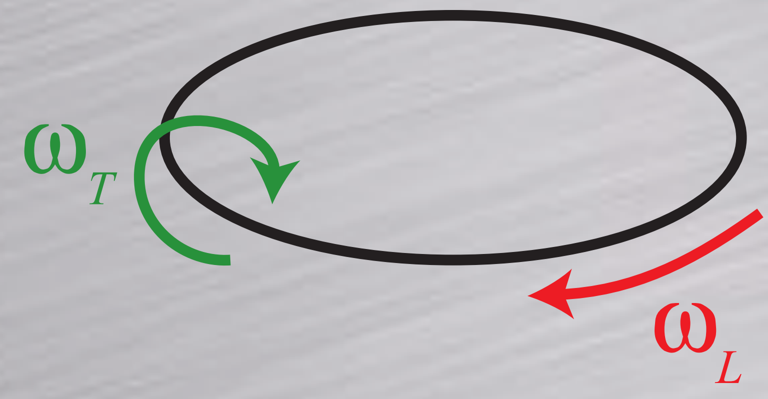
Elliptical Vortex Ring
T.T. Lim (Singapore)



http://serve.me.nus.edu.sg/limtt/#Video_Gallery

Circle Bifurcations

- Elliptic invariant circle has longitudinal & transverse frequencies

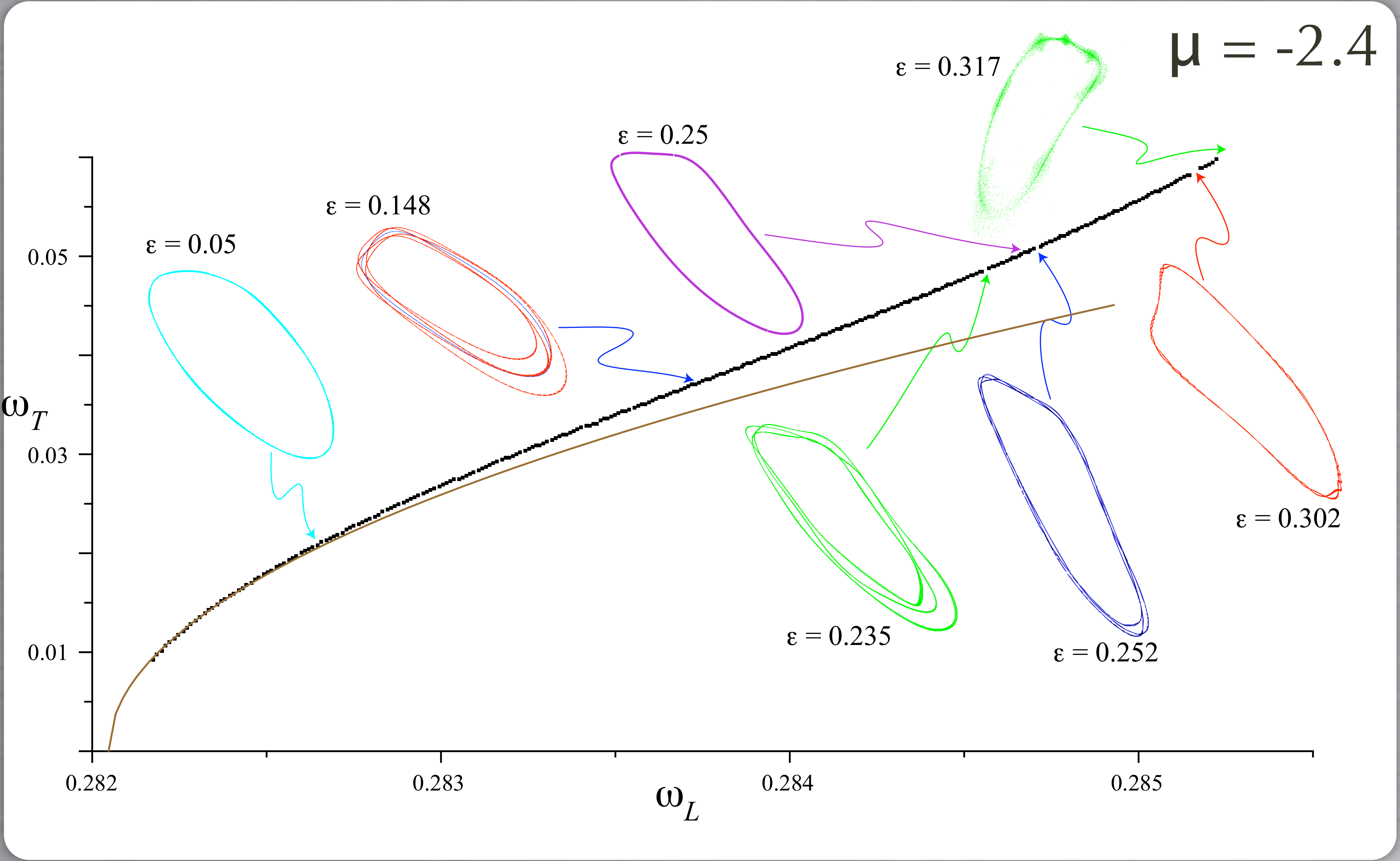


- Bifurcations may occur when

$$m_1\omega_L + m_2\omega_T = k\ , \quad m_1, m_2, k \in \mathbb{Z}$$

Circle Bifurcations

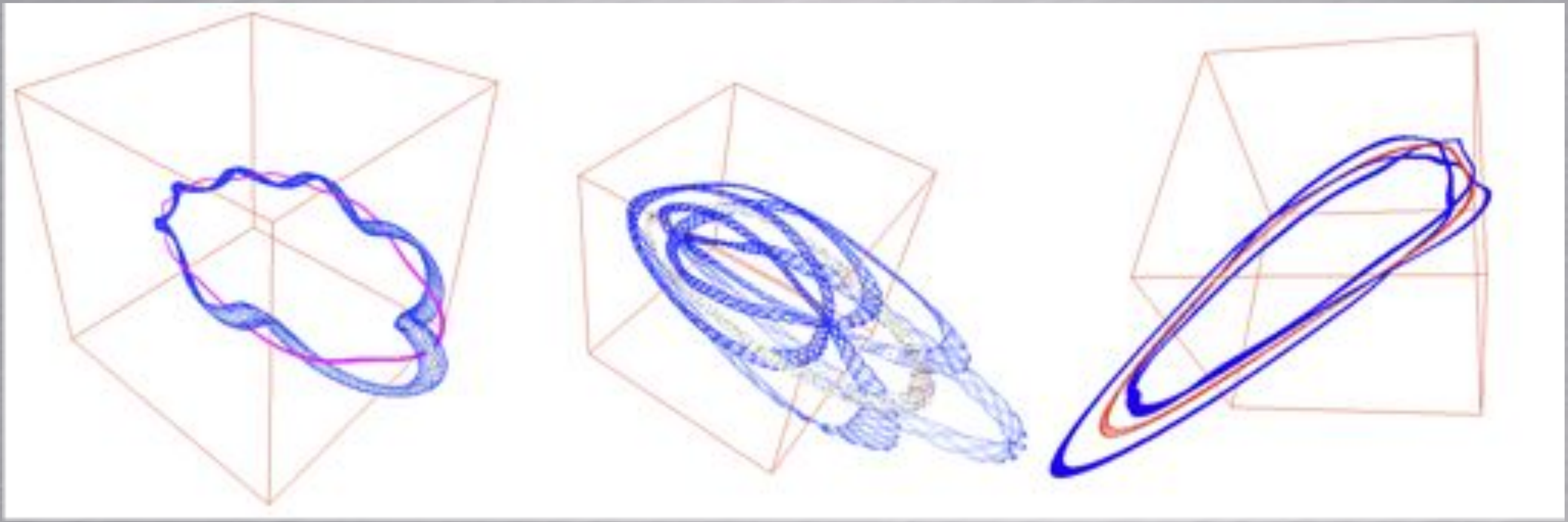
- Circles created in Saddle-Center-Hopf Bifurcation at **ε** = 0



(7,1,2)
(3,4,1)
(3,3,1)
(10,3,3)
(46-2,13)
(7,0,2)

Resonances

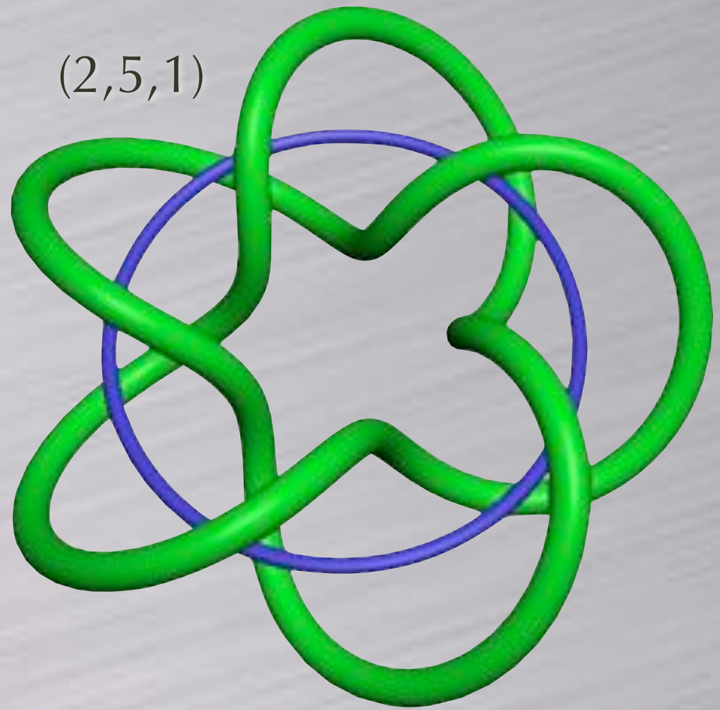
$\mu = -2.4$



(7,1,2) $\varepsilon = 0.052$ (3,5,1) $\varepsilon = 0.1$ (4,-3,1) $\varepsilon = 0.21$

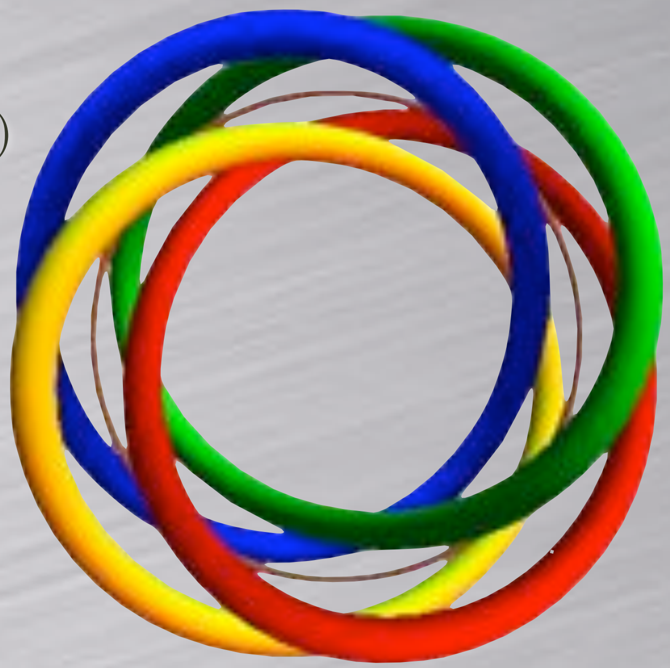
Three types $m.\omega = k$

(2,5,1)

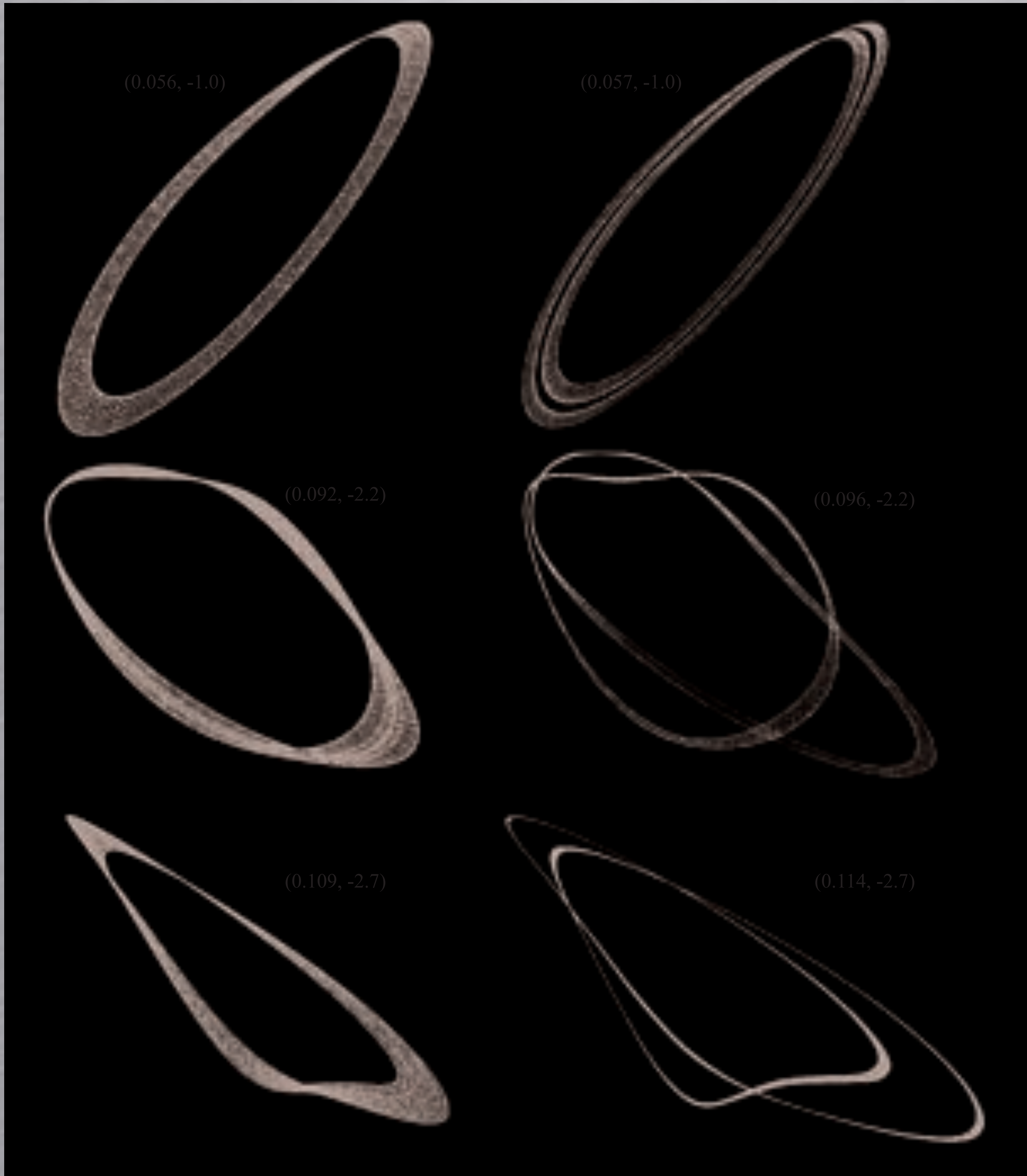


- (m_1,m_2) coprime
  - torus knots
- (m_1,m_2) reducible
  - torus links

(-4,4,1)



- ω_L rational: $(m,k)=(m_1,0,k)$
  - "Pearls on a String"



(1,2,0) Resonance:
Torus Knot

(4,2,1) Resonance:
Torus Link

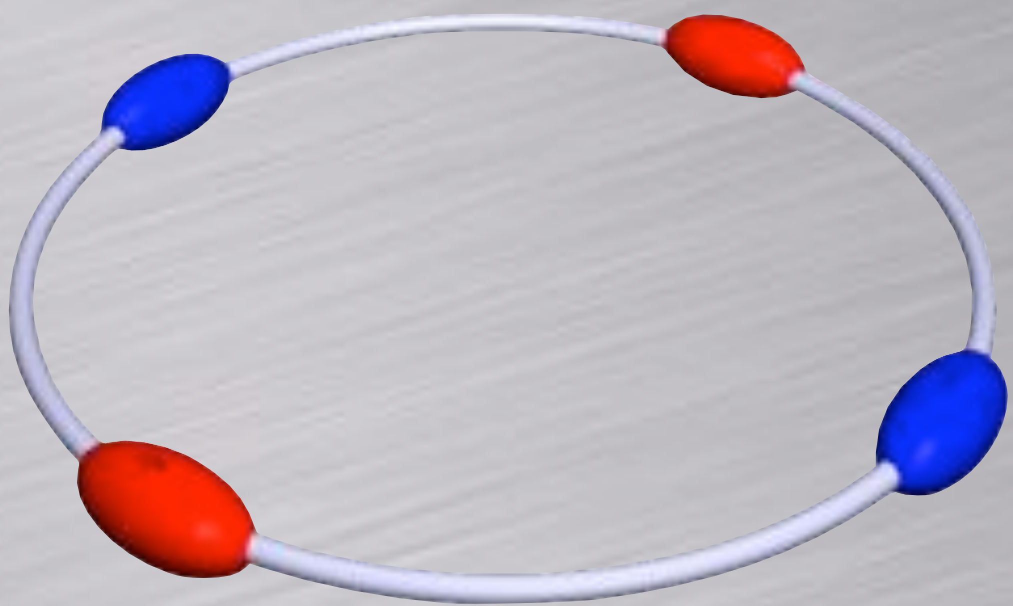
(3,-2,1) Resonance:
Torus Knot

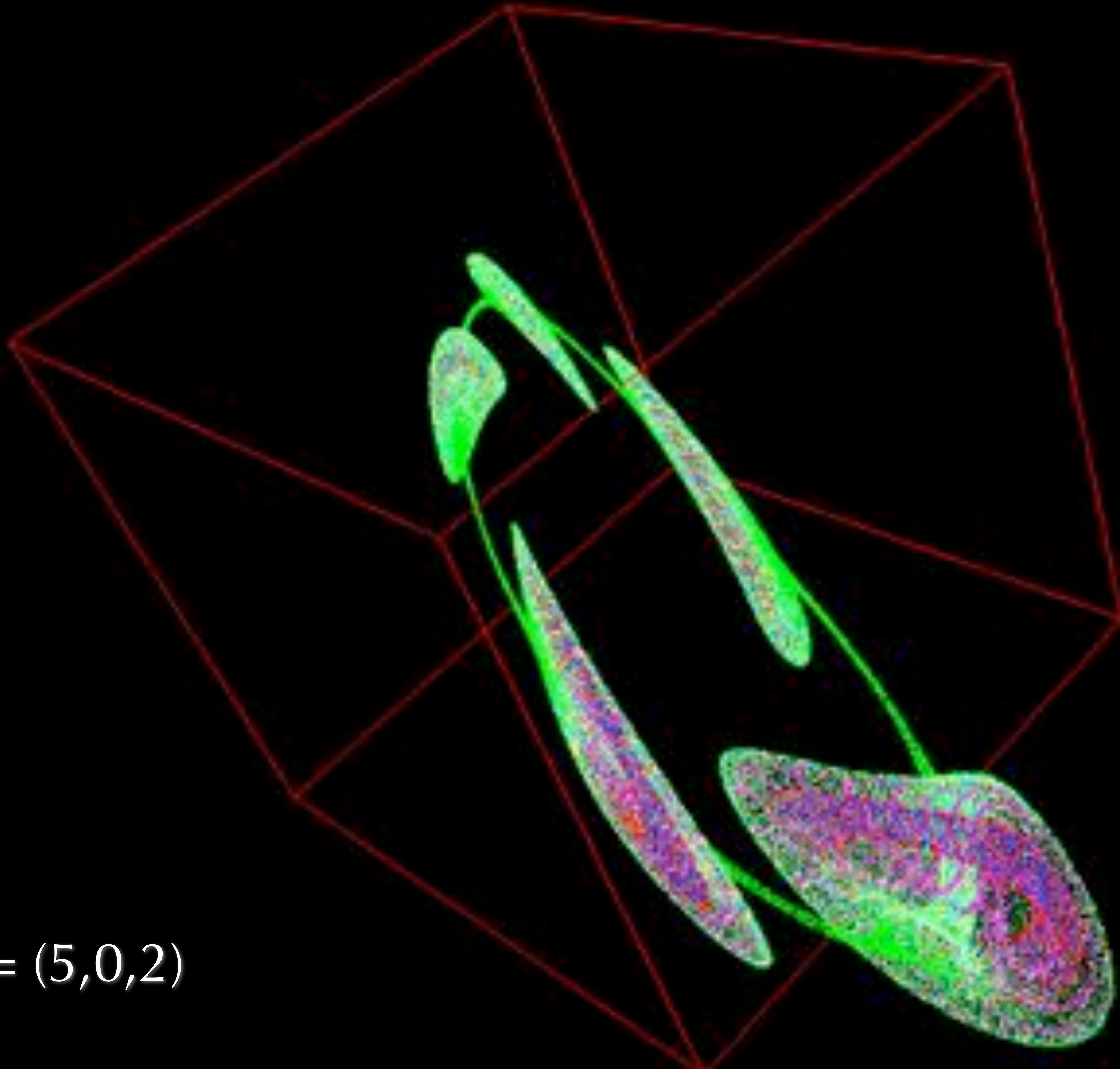
3: Pearls on a String

- Rational ω_L ($m_2 = 0$), the circle mode locks into a pair of periodic orbits
- Generically one is *type-A* and one is *type-B*

(2,0,1)

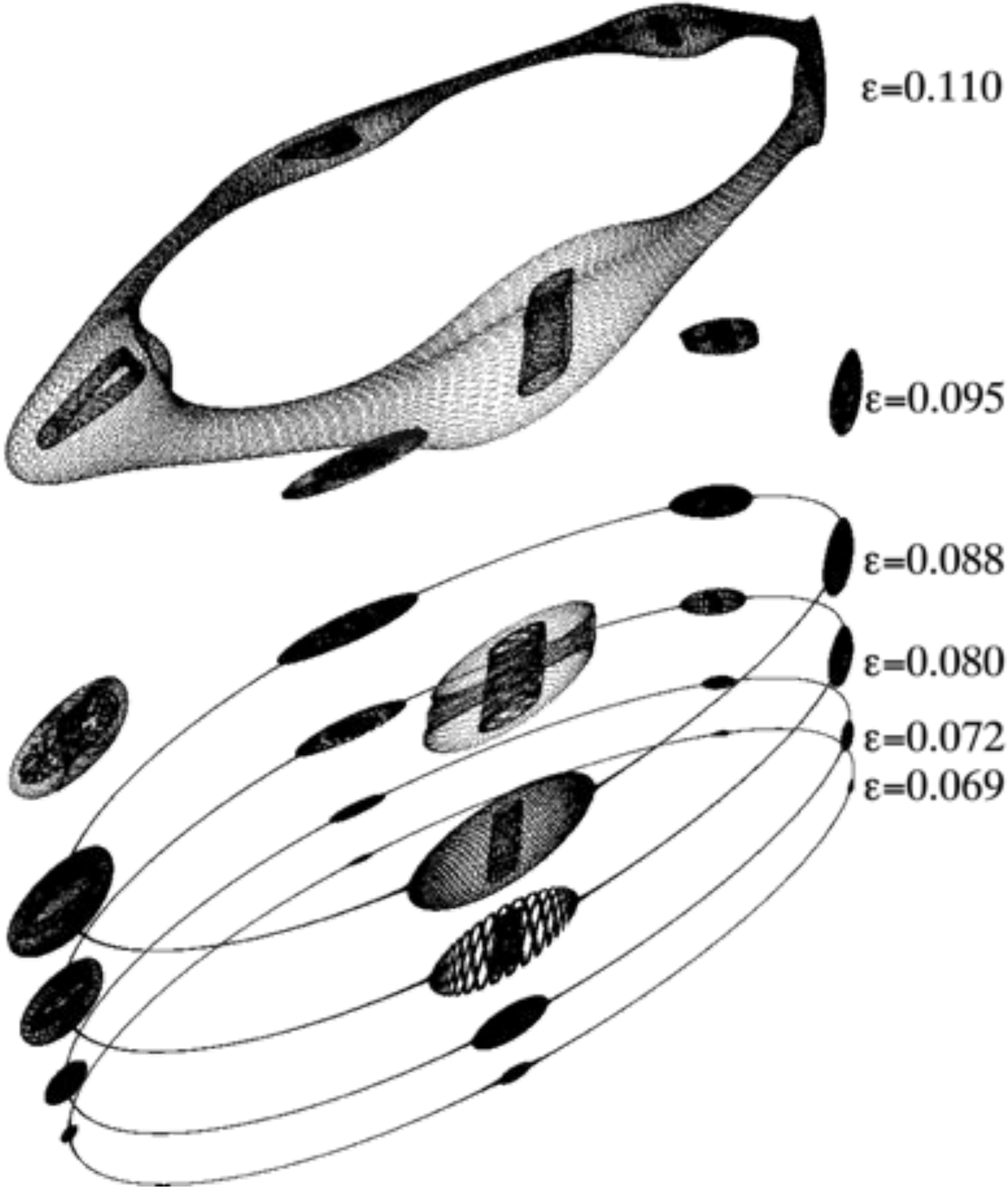
(4,0,1)





$(m,k) = (5,0,2)$

$\mu=-3.64,\ \varepsilon=0.08$

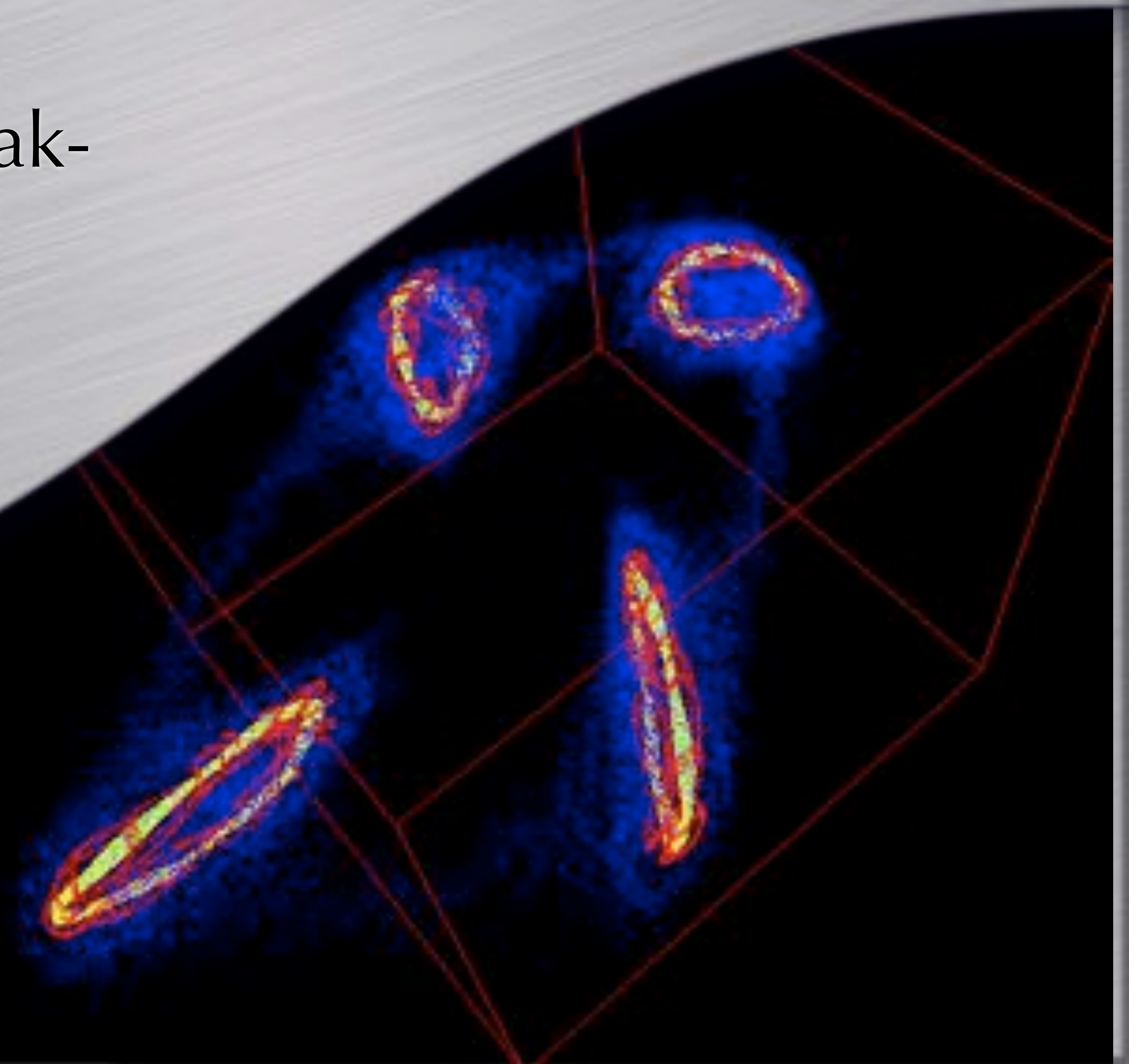


μ = -1.383

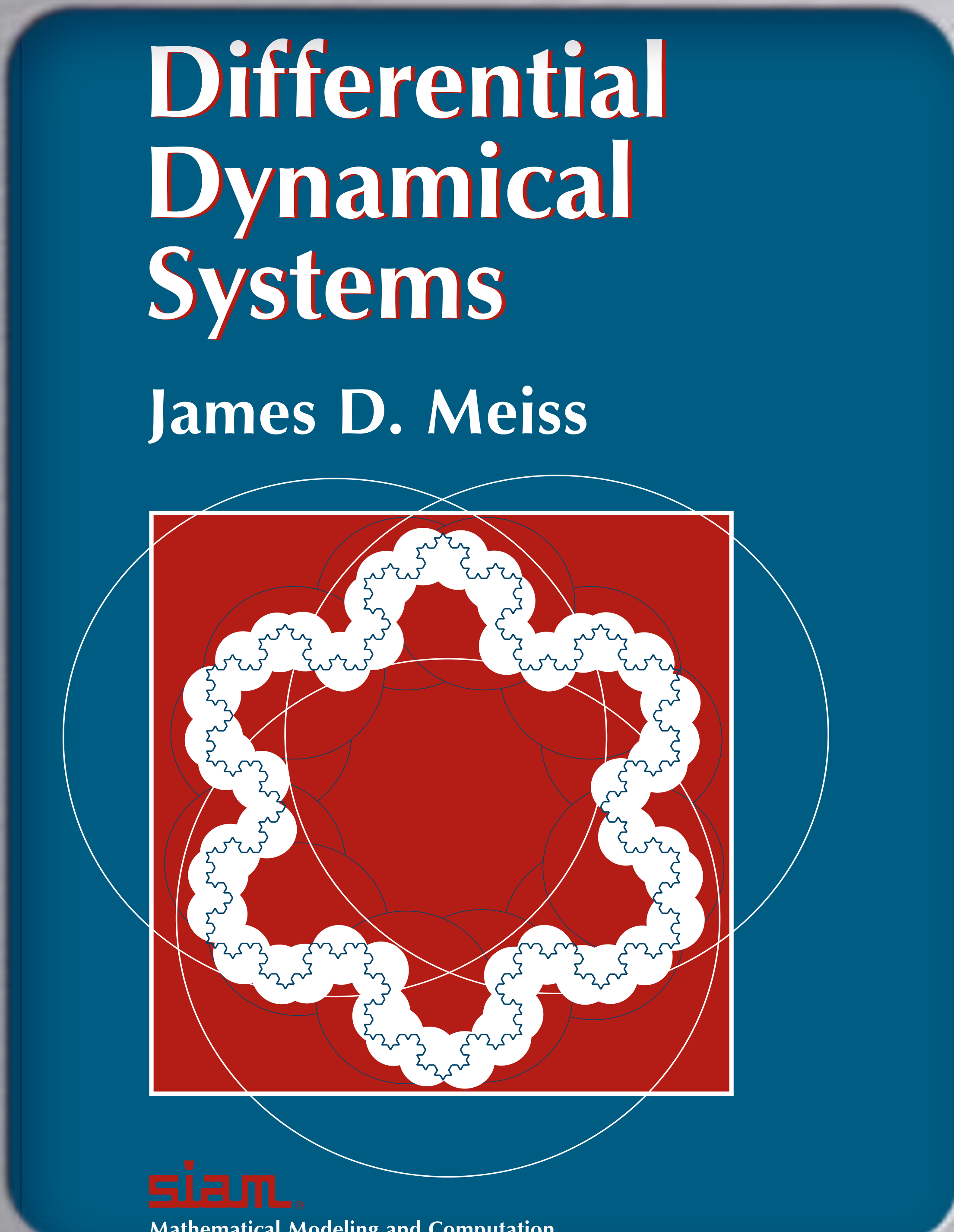
(5,0,1) Pearls

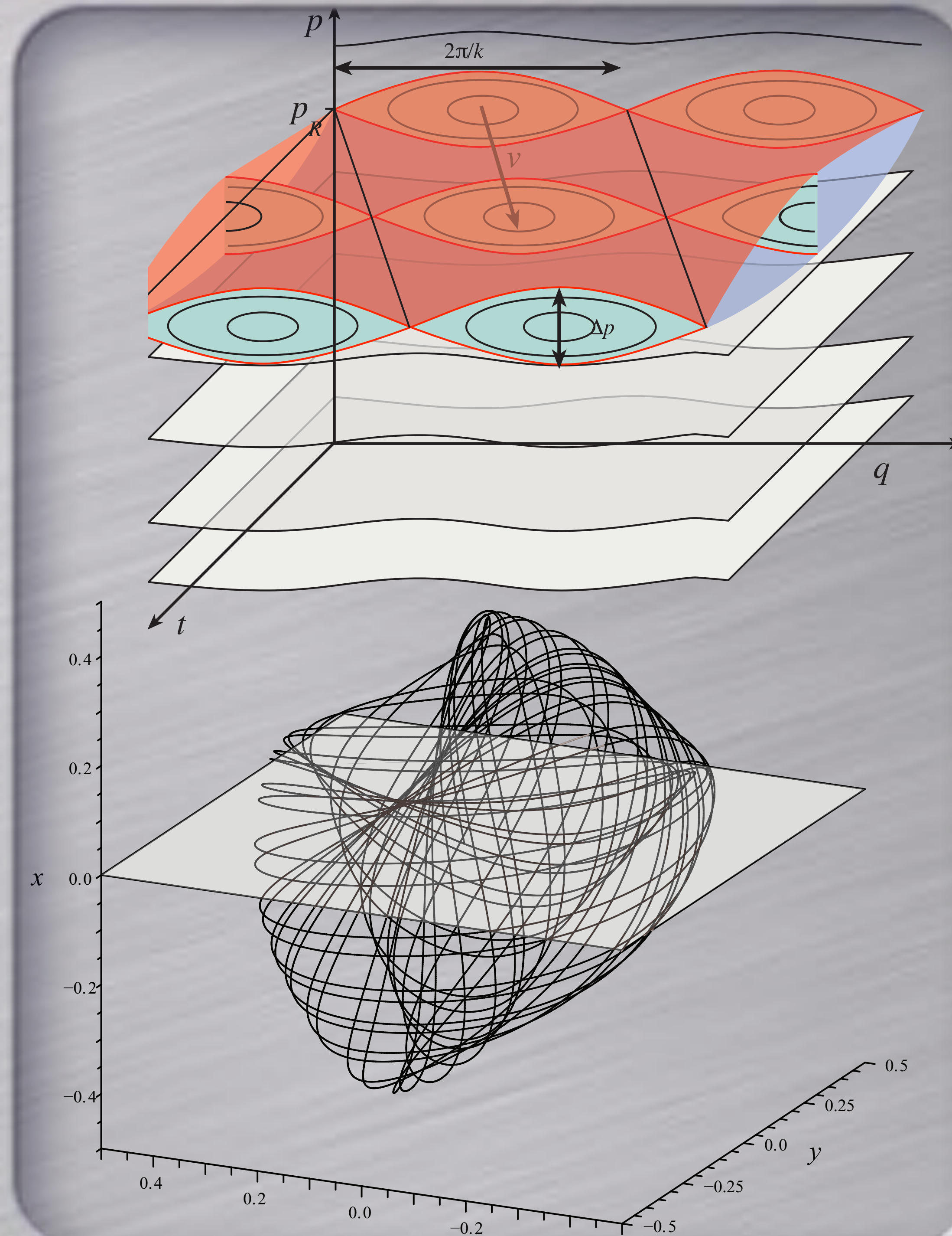
Much still to do...

- Self-Similarity?
- How to tori break-up?
- Last Torus?



Thanks!





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