# Symmetries and Integrability of Volume Preserving Maps

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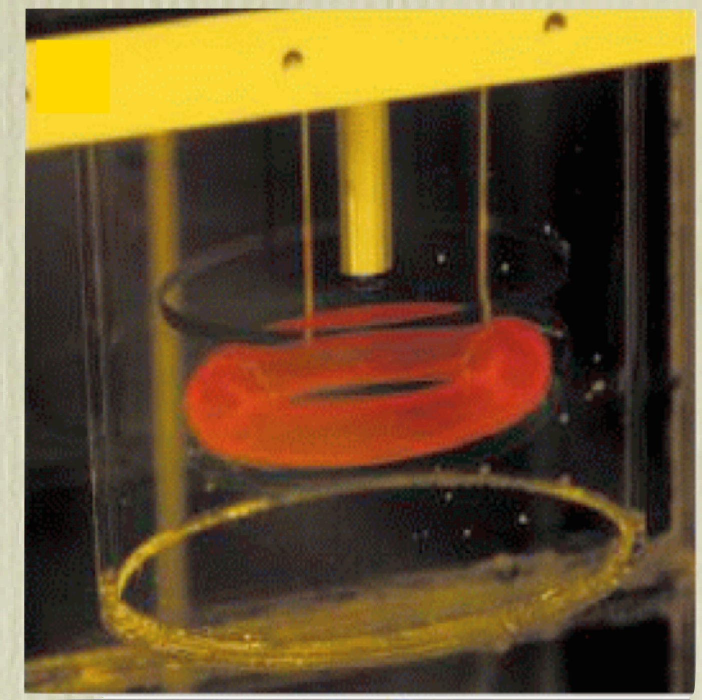
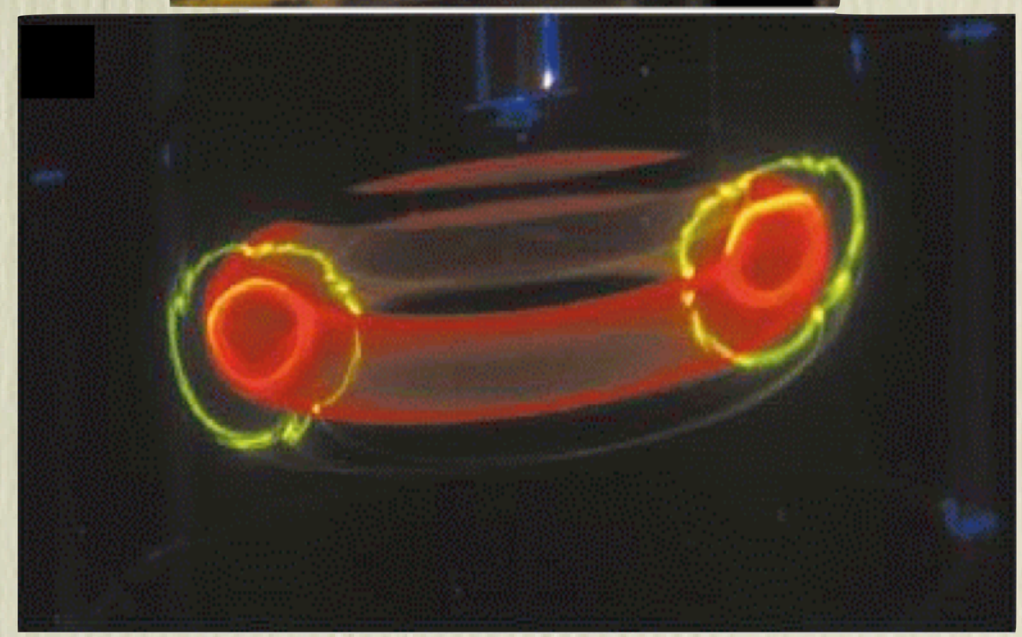
# Why Volume Preserving Maps?

- Simpler than 4D symplectic case
- Mixing (stirring) in incompressible fluids
  - Chaotic advection of dye

$$\dot{x} = v(x,t) \qquad \nabla \cdot v = 0$$

- Magnetic field configurations (with nulls)
  - Earth's magnetotail





- Formally, $f$: $M \rightarrow M$ with volume form $\Omega$:

$$f^{*}\Omega = \Omega$$

$$\Omega = dx_1 \wedge dx_2 \wedge \ldots \wedge dx_n \quad \Rightarrow \quad \det(Df) = 1$$

pullback: $f^{*}\Omega_x(v_1, v_2, v_3) = \Omega_{f(x)}(Df(x)v_1, Df(x)v_2, Df(x)v_3)$

Fountain, et al. (2000). "Chaotic Mixing in a Bounded Three-Dimensional Flow." J. Fluid Mech. 417(265–301).

# Integrability

# Integrability

- Definition?

> It is a well-known fact that for certain problems, auxiliary analytic relations can be deduced by means of which the solutions of the system of differential equations can be satisfactorily treated, in which case the system may be said to be "integrable". When, however, one attempts to formulate a precise definition of integrability, many possibilities appear, each with a certain intrinsic interest.
>
> — G.D. Birkhoff, "Dynamical Systems" §8.13 1927

- Birkhoff: Convergence of formal normal form series near each periodic orbit

- Liouville: Existence of *d* integrals in involution

- Lax Pairs, IST, Painlevé property, etc. = *isospectral integrability*

- Bogoyavlenskij: Symmetries + Integrals = *broad integrability*.

# Liouville Integrability

$\omega = dq \wedge dp$

- Hamiltonian system, $i_{X_H}\omega = dH$, with $d$ degrees of freedom

$$i_{X_H}\omega = X_q dp - X_p dq = \frac{\partial H}{\partial p} dp + \frac{\partial H}{\partial q} dq$$

- $d$ integrals $I_i$: $\quad \frac{d}{dt} I_i = X_H(I_i) = X_H \cdot \nabla I_i = 0$
  - almost everywhere independent: $\nabla I_i \cdot \nabla I_j \neq 0$
  - in involution: $\{I_i, I_j\} = 0$
- then on any compact components of the (regular) integral manifold, $I_i = c_i$, there exist angle-action coordinates $(\theta, \mathcal{J})$ such that

$$X_H = \omega(\mathcal{J}) \cdot \frac{\partial}{\partial \theta}$$

Note: each integral gives a symmetry $\Upsilon$ by $i_\Upsilon \omega = dI$

# Liouville: Symplectic Maps

- symplectic: $f^*\omega = \omega$,    $\omega = dq \wedge dp$ on $2d$ dimensional manifold

$$(q', p') = f(q, p)$$

- $d$ integrals, $f^* I_i = I_i \circ f = I_i$,
- involutory, $\{I_i, I_j\} = 0$, and independent

➡ In the neighborhood of any nonsingular, compact integral manifold there are angle-action coordinates $(\theta, \mathcal{J})$:

$$\theta' = \theta + \omega(\mathcal{J})$$
$$\mathcal{J}' = \mathcal{J}$$

$$\omega(\mathcal{J}) = \nabla S(\mathcal{J})$$

Note: each integral gives rise to symmetry $\Upsilon_i = \partial/\partial\theta_i$

$$f^*\Upsilon = \Upsilon$$

pullback: $f^*\Upsilon(x) \equiv \left(Df(x)\right)^{-1} \Upsilon(f(x))$

Veselov, A. P. (1991). "Integrable Mappings." Russian Math. Surveys 46(5): 1-51.

# 2D Integrable Maps

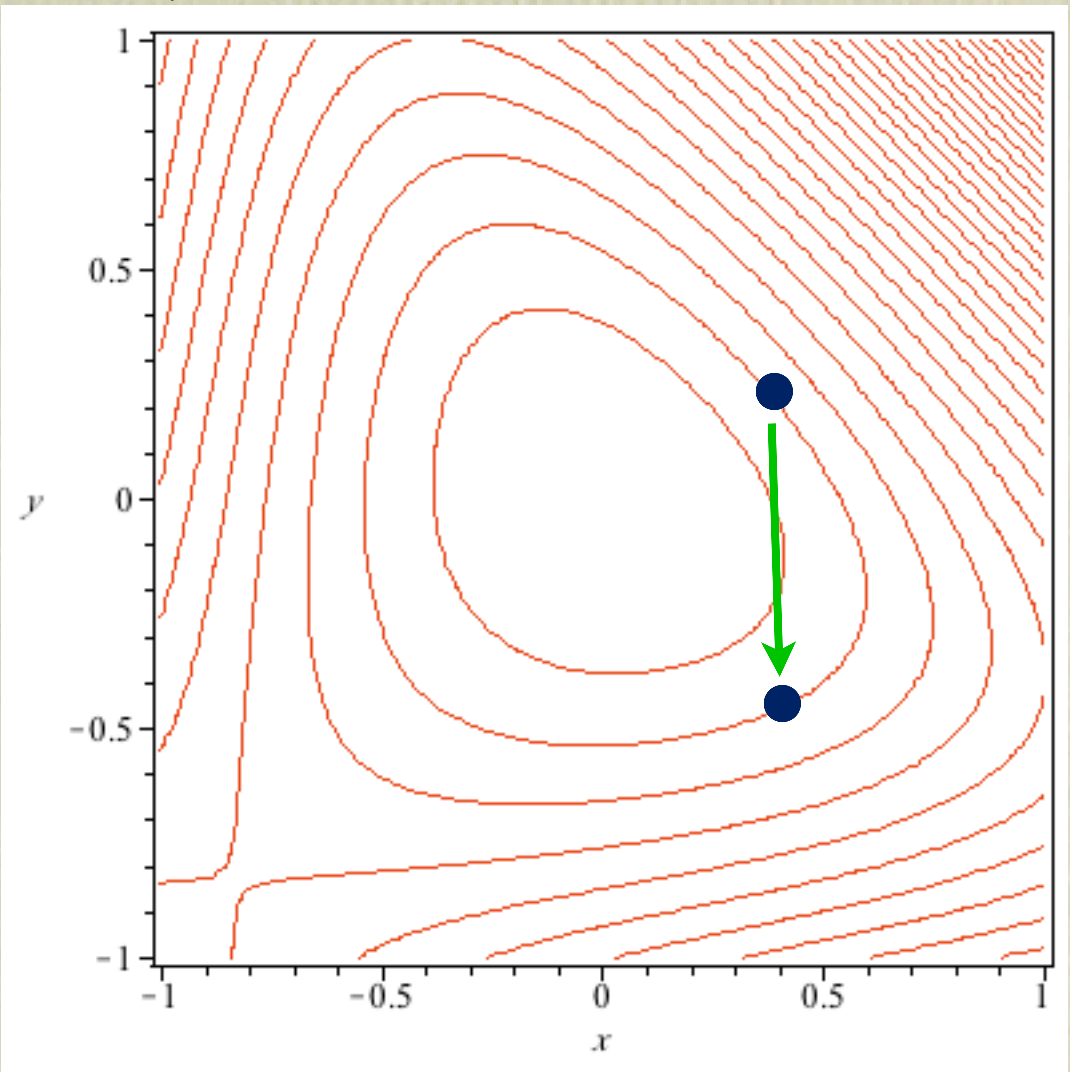
- Many examples
  - McMillan (1971) $f(x,y) = \left( y, -x + \dfrac{by^2 + dy}{ay^2 + by + c} \right)$

$$I = ax^2y^2 + b(x^2y + y^2x) + c(x^2 + y^2) + dxy$$

  - Suris (1989) Maps of Standard type

$$x_{n+2} - 2x_n + x_{n-1} = \varepsilon F(x,\varepsilon)$$

  - QRT Maps (Quispel et al 1988)




a=0,b=c=1,d=1/2


McMillan, E. M. (1971). A Problem in the Stability of Periodic Systems. Topics in Modern Physics, a Tribute to E.V. Condon: 219-244.
Suris, Y. B. (1989). "Integrable Mappings of the Standard Type." Functional Analysis and Applications 23: 74-76.
Quispel, G. R. W., J. A. G. Roberts and C. J. Thompson (1988). "Integrable Mappings and Soliton Equations." Phys. Lett. A 126: 419-421.

# 4D Integrable Maps

- Iatrou examples

$$f(w,x,y,z) = \left(x, y, z, -w - y - g(x+z)\right)$$

$$g(x) = \frac{ax+b}{cx+d}$$

- Symplectic with $\omega = dw \wedge dx + dx \wedge dy + dy \wedge dz$
- Liouville integrable
  - Integrals constructed from a Lax pair
- Or, canonical variables: $\omega = dq \wedge dp$

$$\begin{aligned} q_1' &= -p_1 - g(q_1) \\ q_2' &= -p_1 - p_2 - g(q_1) \\ p_1' &= q_1 - q_2 \\ p_2' &= q_2 \end{aligned}, \quad g(q) = \frac{aq+b}{cq+d}$$

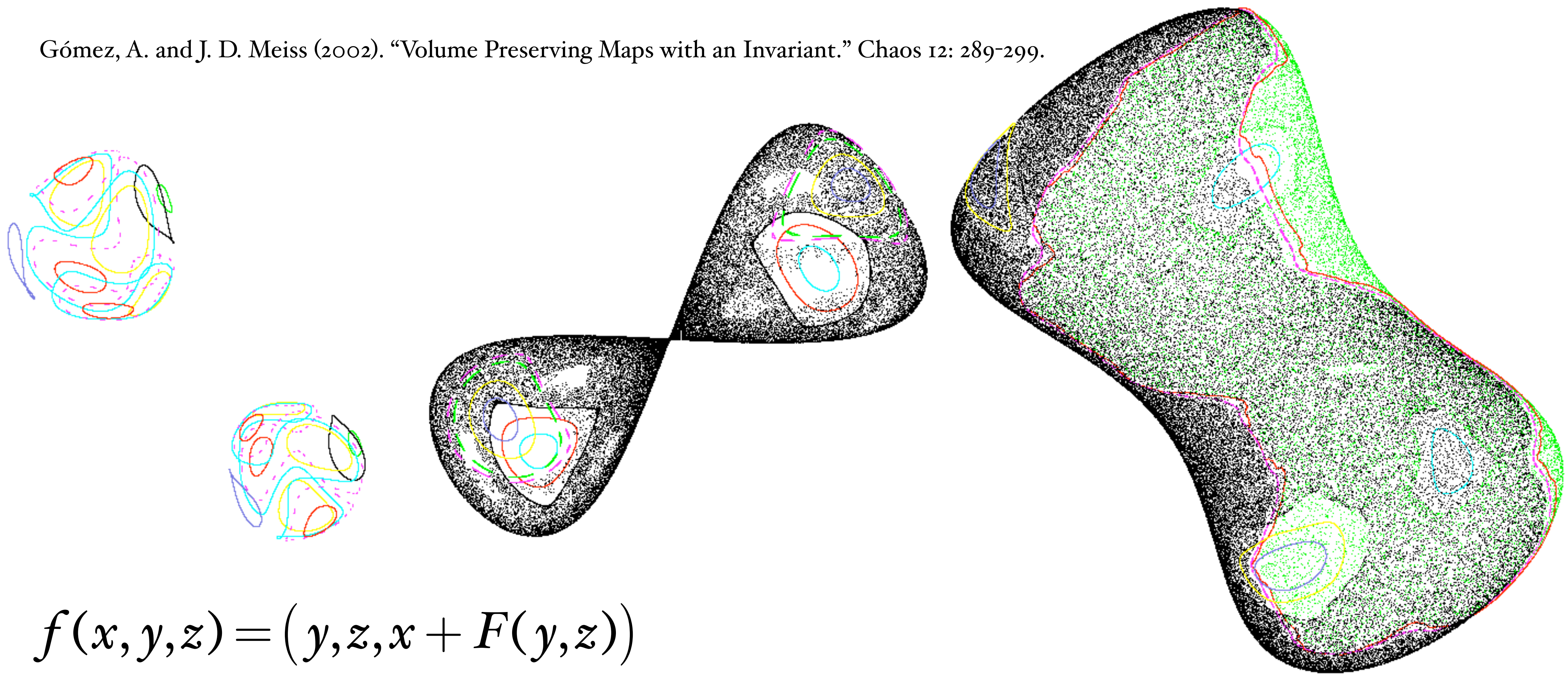

Iatrou, A. (2003). "Higher Dimensional Integrable Mappings." Physica D 179: 229-253.
McLachlan, R. I. (1993). "Integrable Four-Dimensional Symplectic Maps of Standard Type." Phys Lett A 177(3): 211-214.

# 3D Volume Preserving Maps: One or Two Integrals?

# One Integral is Not Enough

- $I$ is an integral if $I \circ f = I$
- For 2D maps, Integral ⇒ Integrable
- For 3D Volume preserving maps ⇒ $f$ reduces to A.P. map on contours

Gómez, A. and J. D. Meiss (2002). "Volume Preserving Maps with an Invariant." Chaos 12: 289-299.



$$f(x,y,z) = \big(y, z, x + F(y,z)\big)$$

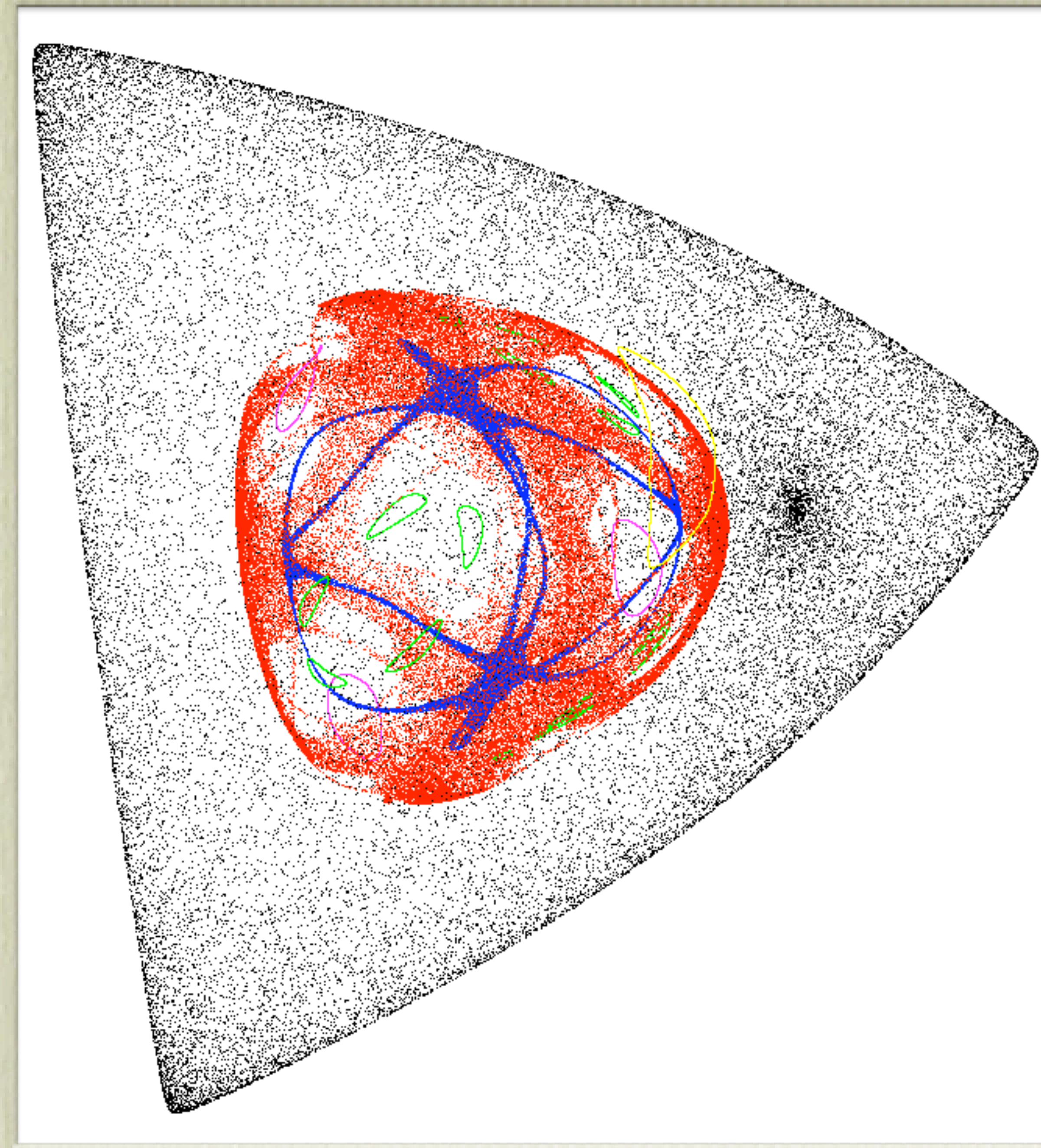
$$F(y,z) = \frac{(y-z)(\alpha - \beta yz)}{1 + \gamma(y^2 + z^2) + \beta yz + \delta y^2 z^2}$$

$$(\alpha, \beta, \gamma, \delta) = (2, -2, 1, 0)$$

Roberts, J. and M. Baake (1994). "Trace Maps as 3D Reversible Dynamical Systems with an Invariant." Journal of Statistical Physics 74(3): 829-888.

# Trace Maps

Dynamics semi-conjugate to Anosov on outermost ($I$=0) bounded surface.



$$f(x,y,z) = (-y + 2xz, z, -x - 2yz + 4xz^2)$$

$$I = x^2 + y^2 + z^2 - 2xyz - 1$$

# Two Integrals are Too Many?

- ∃ 2 Integrals (locally independent) ⇒ Orbits confined to curves
- Example: 3D Lyness Map (2D version by Lyness in 1945)

$$f(x,y,z) = \left( y, z, \frac{a+y+z}{x} \right)$$

- Volume reversing $f^*\Omega = -\Omega$ $\qquad \Omega = \frac{1}{xyz} dx \wedge dy \wedge dz$

$$I_1 = \frac{(a+x+y+z)(1+x)(1+y)(1+z)}{xyz}$$

$$I_2 = \frac{\big(a+x+y+z(1+x)\big)(1+x+y)(1+y+z)}{xyz}$$

- so one definition of *integrable*: ∃ $n$-1 integrals. However, the dynamics is *trivial* in that case!

Cima, A., A. Gasull and V. Mañosa (2008). "Some Properties of the $k$D Lyness Map." J. Phys A 41: 285205.

Also: Iatrou (arXiv 0306052) for Gomez-like 3D examples with 2 integrals

# KAM Theory for V.P. Maps

- integrable one-action maps: $M = \mathbb{T}^2 \times \mathbb{R}$ + perturbation

$$\left.\begin{aligned} \theta_1' &= \theta_1 + \omega_1(J) \\ \theta_2' &= \theta_2 + \omega_2(J) \\ J' &= J \end{aligned}\right\} + O(\varepsilon)$$

- integrable two-action maps: $M = \mathbb{T} \times \mathbb{R}^2$ + perturbation

$$\left.\begin{aligned} \theta' &= \theta + \omega(J_1, J_2) \\ J_1' &= J_1 \\ J_2' &= J_2 \end{aligned}\right\} + O(\varepsilon)$$

- one-action, KAM results of Cheng & Sun and Xia require
  - exact volume-preserving perturbations (intersection property)
  - nondegeneracy of frequency map or additional parameters


Xia, Z. (1992). "Existence of Invariant Tori in Volume-Preserving Diffeomorphisms." Erg. Th. Dyn. Sys. 12(3): 621-631.
Cheng, C.-Q. and Y.-S. Sun (1990). "Existence of Invariant Tori in 3D Measure-Preserving Mappings." Celestial Mech. 47(3): 275-292.

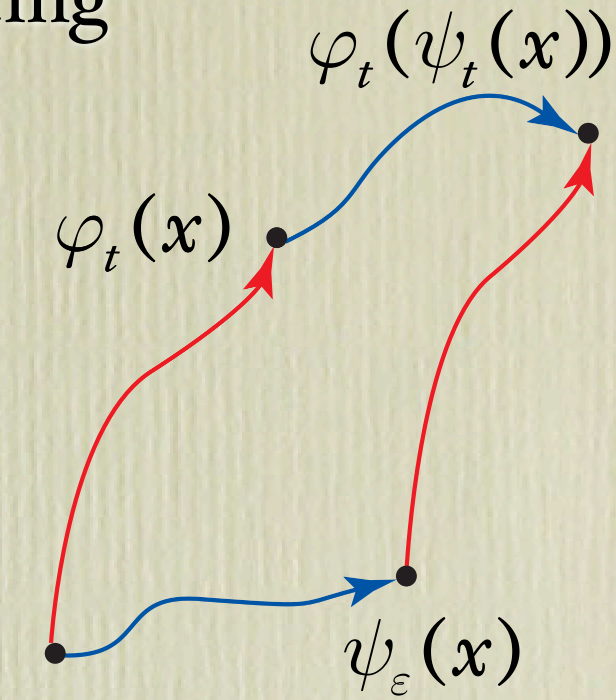
# Symmetry

# Symmetry

- Vector field $X$ has symmetry generated by $\Upsilon$ if commuting

$$[X,\Upsilon] = X\cdot\nabla\Upsilon - \Upsilon\cdot\nabla X = 0$$

➡Flow $\psi_\varepsilon(x)$ of $\Upsilon$ commutes with flow $\varphi_L(x)$ of $X$




- Map $f$ has symmetry $\Upsilon$ if $\quad f^*\Upsilon = \Upsilon, \quad \Rightarrow \quad \psi_\varepsilon \circ f = f \circ \psi_\varepsilon$

- Sometimes Symmetry ⇒ Integral, as in Noether's theorem

$$I = i_\Upsilon\left(\frac{\partial L}{\partial \dot q}dq\right)$$

- or Bernoulli's theorem...

# Bernoulli: Symmetry ⇒ Integral

- $X$ incompressible ⇒ $\nabla\cdot X = 0$
- $Y$ incompressible symmetry ⇒ $\nabla\cdot Y = 0$ & $[X,Y] = 0$,
- Integral
    - $\nabla I = X\times Y$ $\longrightarrow$ $\dfrac{dI}{dt} = X\cdot\nabla I = X\cdot X\times Y = 0$

    ➡ $I$ exists since $\nabla\times X\times Y = [X,Y] + (\nabla\cdot Y)X - (\nabla\cdot X)Y = 0$

An example is the vorticity $\nabla\times v$ for an incompressible, stationary fluid

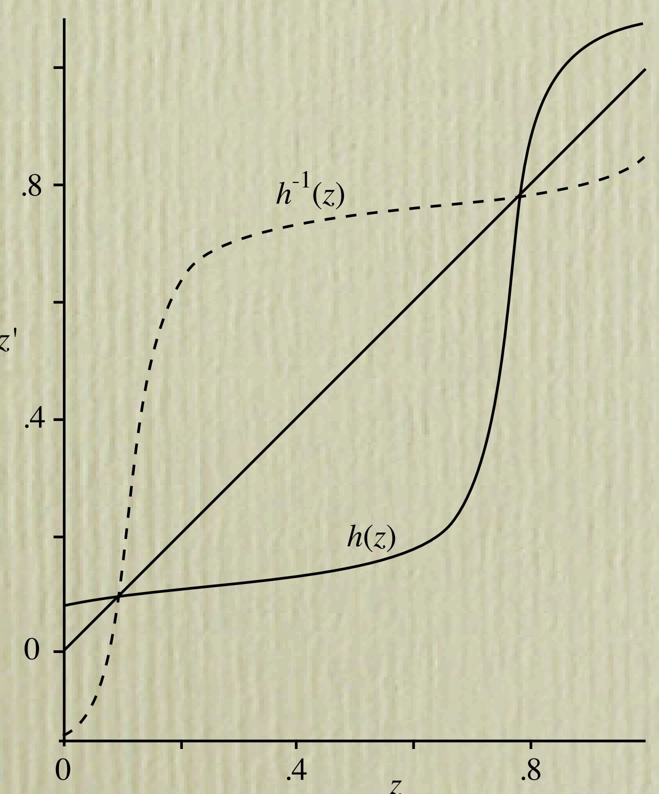
- More
    - $\Omega = d\alpha$ exact volume form
    - incompressible: $i_X\Omega = 0$ ⇒ $L_X\,\alpha = d\beta_X$ (generating form)
    - incompressible symmetry $Y$ ⇒ $L_Y\,\alpha = d\beta_Y$ & $[X,Y] = 0$

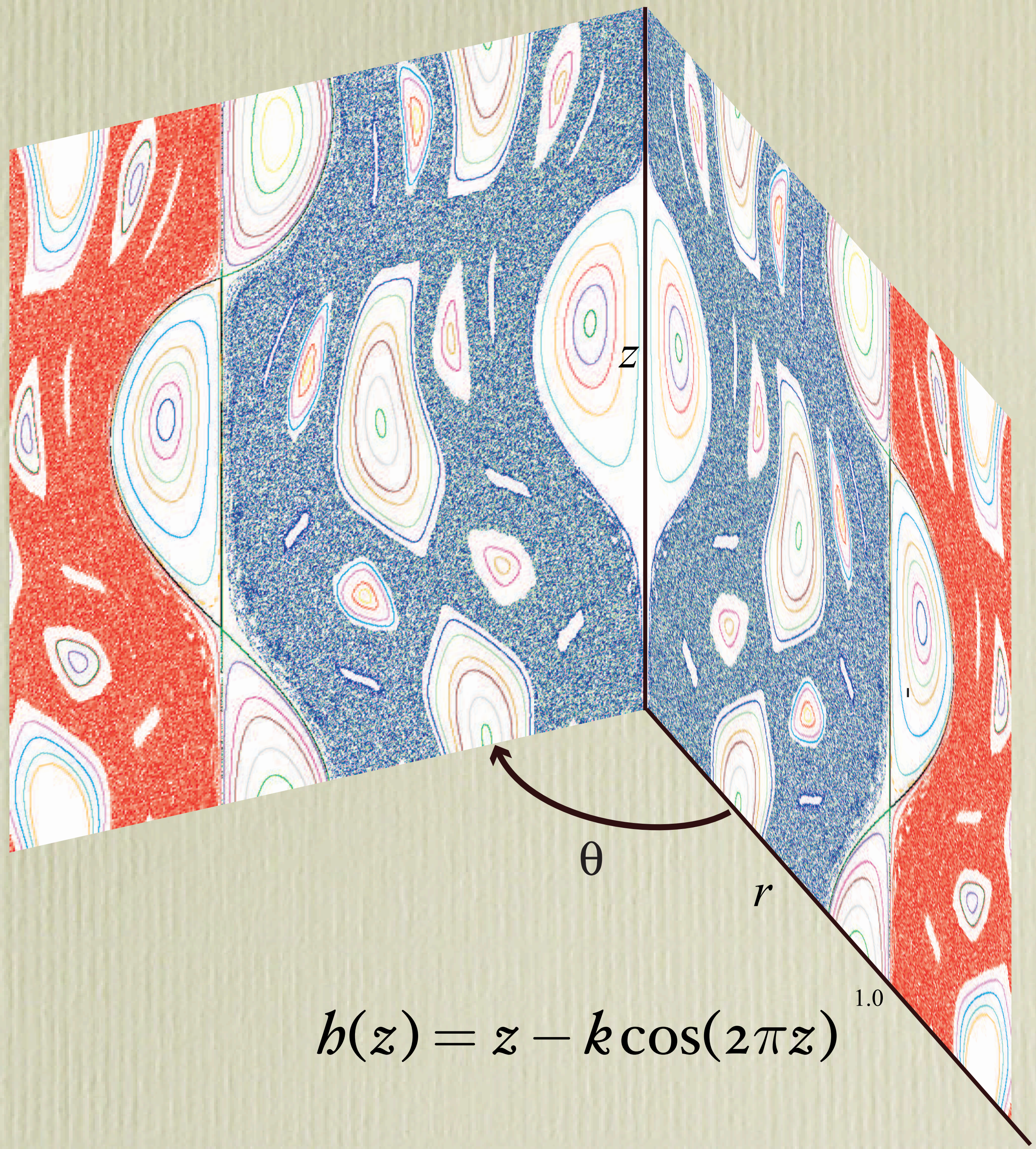
    ➡ Integral $I = i_X i_Y\,\alpha + i_X\beta_Y - i_Y\beta_X$

# Symmetry $\not\Rightarrow$ Integral

- Suppose, e.g. $f$ commutes with a rotation $\Upsilon = \dfrac{\partial}{\partial\theta}$

$$r' = h^{-1}(r + h(z)) - z$$
$$z' = h(z) + r - 1$$
$$\theta' = \theta + \rho(r,z)$$




$$h(z) = z - k\cos(2\pi z)$$

Lomelí, H. E. and J. D. Meiss (2003). "Heteroclinic Intersections between Invariant Circles of Volume-Preserving Maps." Nonlinearity 16(5): 1573-1595.

# Symmetry Reduction

- If $f$ has symmetry $\Upsilon$, then

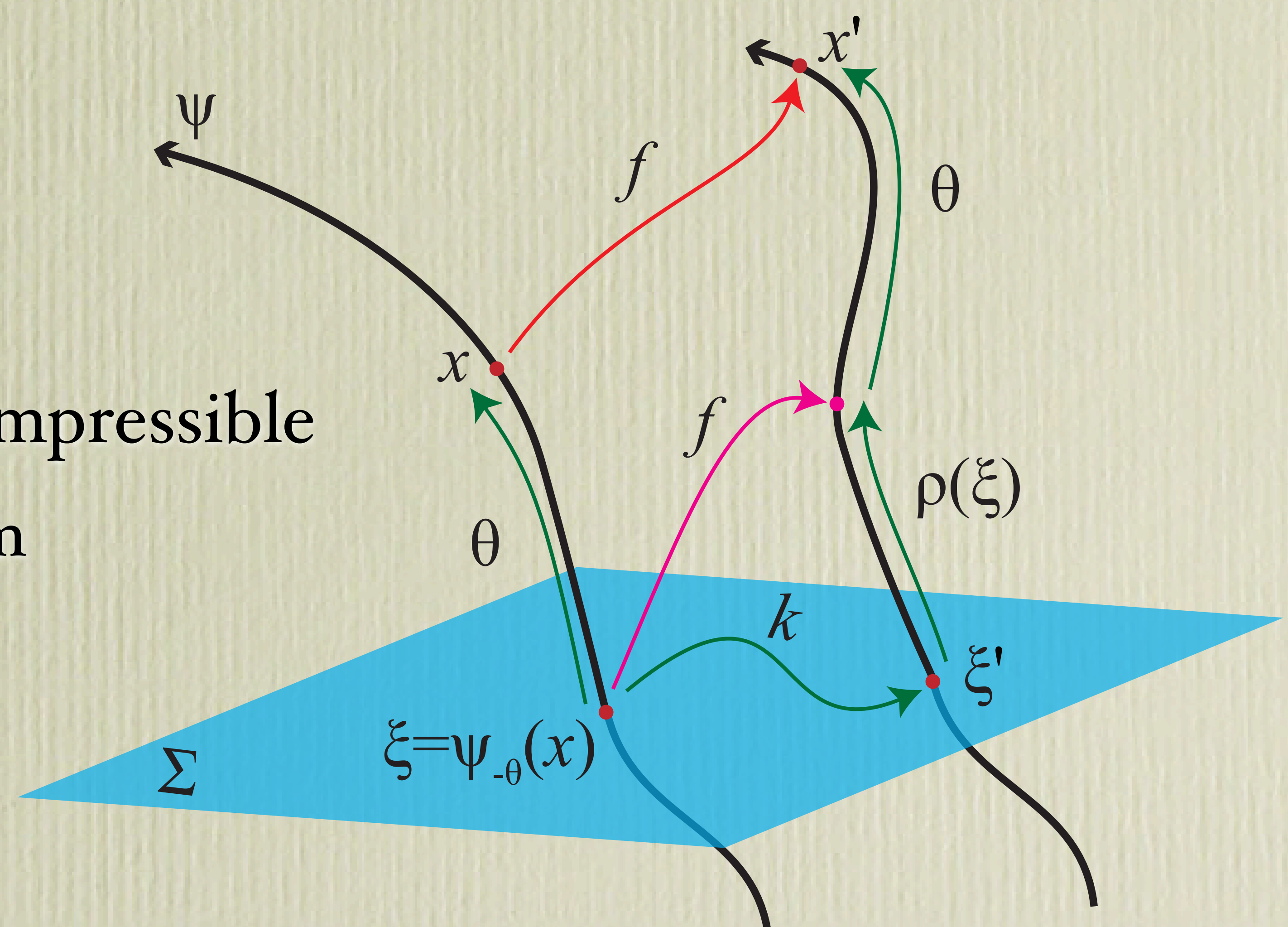
$$f^*\Upsilon = \Upsilon \quad \Rightarrow \quad f \circ \psi_\theta = \psi_\theta \circ f$$

  - locally: section $\Sigma$ of $\Upsilon$
  - projection $\pi$: $M \longrightarrow \Sigma$ using flow

$$x = \psi_\theta(\xi) \quad x' = \psi_{\theta'}(\xi')$$

    ➡ reduction to skew product form on $\Sigma \times [a,b]$

$$\xi' = k(\xi)$$
$$\theta' = \theta + \rho(\xi)$$




- $f$ volume-preserving & $\Upsilon$ incompressible

    ➡ $k$: $\Sigma \rightarrow \Sigma$ is V.P. with form

$$\omega = i_\Upsilon \Omega|_\Sigma$$

# Example

- Hopf-Saddle-Node Normal form

$$x' = x + y$$
$$y' = y + z'$$
$$z' = z + p(x, y)$$

Dullin, H. R. and J. D. Meiss (2008). “Nilpotent Normal Forms for a Divergence-Free Vector Fields and Volume-Preserving Maps.” Phys. D 237(2): 155-166

- exact, V.P. diffeomorphism for any smooth *p*
- Symmetry (only?) if $p(y)$ alone
    - $\psi_\theta(x,y,z) = (x+s,y,z)$
    - $(y,z)$ dynamics is generalized standard map
    - $\varrho(y,z) = y$

# Symmetries are not enough

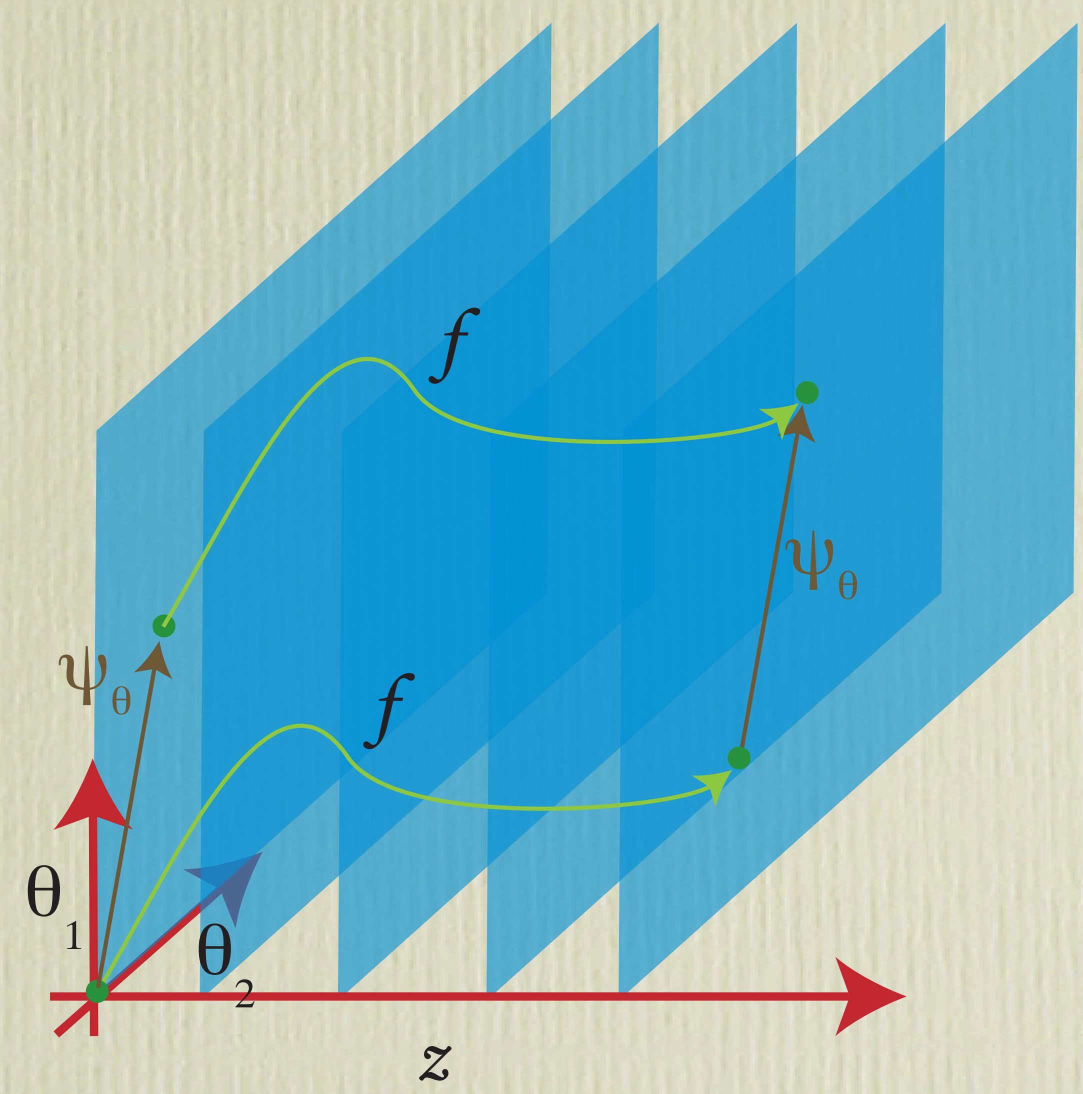
- $\mathbb{T}^2 \times \mathbb{R}$

$$\theta_1' = \theta_1 + \rho_1(z)$$
$$\theta_2' = \theta_2 + \rho_2(z)$$
$$z' = h(z)$$

$h$ any homeomorphism

- Two symmetries: $\Upsilon_i = \dfrac{\partial}{\partial \theta_i}$

- No invariants
  - e.g., $h$ could be itself a circle map and orbits dense on $\mathbb{T}^3$



# Integrals and Symmetry?

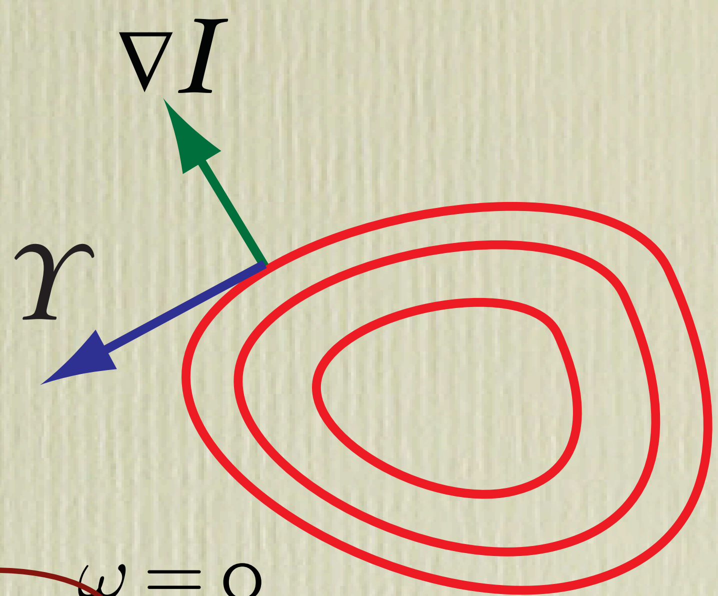
# Integral + Symplectic ⇒ Symmetry

- Anti-Noether Lemma:

  $f$ symplectic, $f^*\omega = \omega$ & has an integral, $I$,

  ➡ Hamiltonian vector field $\Upsilon$:

$$i_\Upsilon \omega = dI$$




is symmetry

$$f^*(i_\Upsilon \omega) = i_{f^*\Upsilon} f^*\omega = i_{f^*\Upsilon} f^*\omega$$
$$f^* dI = d(I \cdot f) = dI = i_\Upsilon \omega$$
$$\Rightarrow i_{f^*\Upsilon - \Upsilon}\omega = 0$$

ω nondegenerate

- Example: McMillan Map

$$f(x,y) = \left( y, -x + \frac{bx^2 + dx}{ax^2 + bx + c} \right)$$

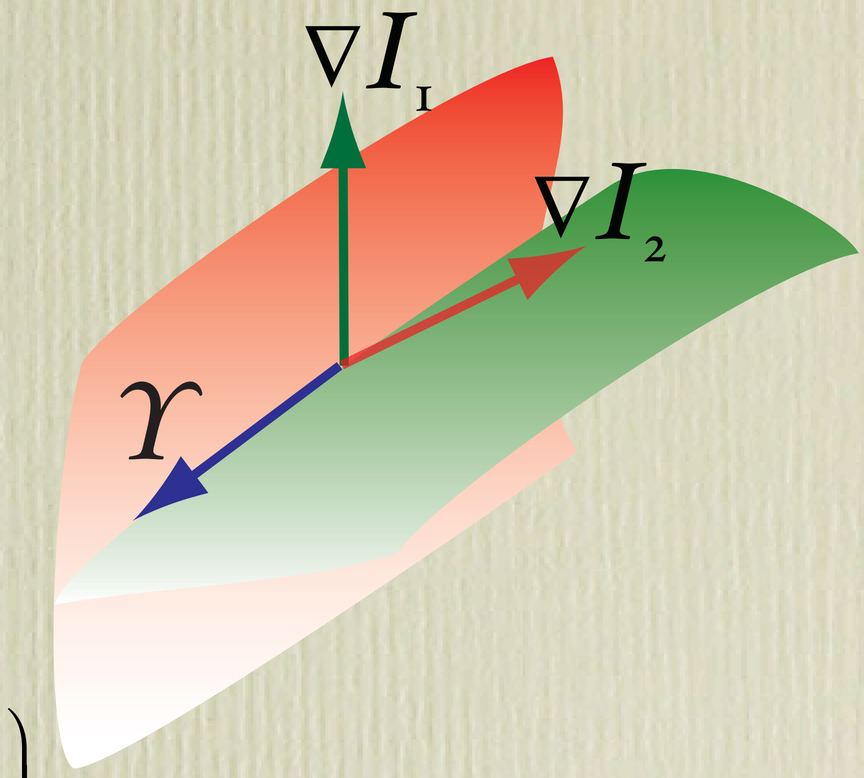
$$I = ax^2y^2 + b(x^2y + y^2x) + c(x^2 + y^2) + dxy$$

➡ $$\Upsilon = \begin{bmatrix} 2Ax^2y + 2Bxy + Bx^2 + 2Cy + Dx \\ -2Axy^2 - 2Bxy - By^2 - 2Cx - Dy \end{bmatrix}$$

# Two Integrals+V.P ⇒ Symmetry

- $f^*\Omega = \Omega$
- $f^*I_i = I_i$, $i = 1, 2$
- The restriction of $f$ to a surface $I_1 = c$ is symplectic with form

$$\omega = \frac{i_{\nabla I_1}\Omega}{|\nabla I_1|^2}$$




- thus Hamiltonian vector field $\Upsilon$ for $I_2$,

$$i_\Upsilon \omega = dI_2$$

  is a symmetry of $f$

- Example: Lyness map $f(x,y,z) = \left(y, z, \frac{a+y+z}{x}\right)$

$$\Upsilon = \frac{1}{xyz}\begin{bmatrix} x(x+1)(1+y+z)(a+x+y-yz) \\ y(y+1)(x-z)(a+x+y+z+xz) \\ -z(z+1)(1+x+y)(a+y+z-xy) \end{bmatrix}$$

# Broad Integrability

# Broad Integrability: Vector Fields

- A vector field $X$ on an $n$-manifold $M$ is *broadly integrable* if there are
  - $n$-$k$ integrals $I_i$ (almost every where independent, compact invariant sets); and
  - $k$ symmetries: commuting vector fields $Y_j$ ($[X,Y_i] = 0$, $[Y_j,Y_k] = 0$), preserving $I_i$: ($L_{Y_j}(I_i) = 0$ ).

- Or geometrically if there is a manifold $P$ of dimension $n$-$k$ and neighborhood $U$ of each point in $P$ such that
  - *a fibration $\pi : M \longrightarrow P$ with fibers of dimension $k < n$ that are compact, connected and invariant under the flow of $X$;*
  - *a locally free, infinitesimal action of $\mathbb{T}^k$ on $\pi^{-1}(U)$ that leaves $X$ and the fibers of $\pi$ invariant .*

- Then the fibers of $\pi$ are diffeomorphic to $\mathbb{T}^k$ and there exist angle-action coordinates $(\theta, \mathcal{J})$ on $\pi^{-1}(U) \cong \mathbb{T}^k \times \mathbb{R}^{n-k}$ so that $X$ becomes

$$X = \omega(\mathcal{J}) \cdot \frac{\partial}{\partial \theta}$$


Bogoyavlenskij, O. J. (1998). "Extended Integrability and Bi-Hamiltonian Systems." CMP 196: 19-51.
Fassò, F. and A. Giacobbe (2002). "Geometric Structure of 'Broadly Integrable' Hamiltonian Systems." J. Geom. Phys. 44: 156-170.

# Broad Integrability: Maps

- 3D V.P. map

  1 Symmetry + 1 integral ⇒ *integrable*

- symmetry $\Upsilon$ & integral $I$
  - $I$ invariant under $\Upsilon$ too: $\Upsilon(I) = \Upsilon \cdot \nabla I = 0$
    - ➡ Reduced map has integral $I$ as well
    - ➡ orbits of $\xi' = k(\xi)$ lie on contours of $I$
    - ➡ Since $\Sigma$ is a section for $\Upsilon$, contours of $I$ have transverse intersection
  - 2D Map on $\Sigma$ is symplectic & integrable in the traditional sense

  - $n$ dimensional case:
    - ➡ *k symmetries*, $\Upsilon_i$, $f^*\Upsilon_i = \Upsilon_i$, $[\Upsilon_i, \Upsilon_j] = 0$
    - ➡ *n-k integrals*, $I_j$, invariant, $\Upsilon_i \cdot \nabla I = 0$

$$\theta_i' = \theta_i + \rho_i(I)$$

$$I_j' = I_j$$

# What does this have to do with the price of bread?

- Not really sure....

- Understanding integrable maps essential for KAM theory.
- Develop a "canonical" perturbation theory for V.P. case?
- Have an excuse to come to Mexico and get out of the snow?