Quadratic volume preserving maps: an extension of a result of Moser

Kathryn E. Lenz
Department of Mathematics and Statistics
University of Minnesota
Duluth, MN 55812

Héctor E. Lomelí
Department of Mathematics
Instituto Tecnológico Autónomo de México
México, DF 01000

James D. Meiss*
Department of Applied Mathematics
University of Colorado
Boulder, CO 80309

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Dedicated to Jürgen Moser on the occasion of his 70^{th} *birthday*

Abstract

A natural generalization of the Hénon map of the plane is a quadratic diffeomorphism that has a quadratic inverse. We study the case when these maps are volume preserving, which generalizes the the family of symplectic quadratic maps studied by Moser. In this paper we obtain a characterization of these maps for dimension four and less. In addition, we use Moser's result to construct a subfamily of in n dimensions.

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1 Introduction

Some of the simplest nonlinear systems are given by quadratic maps: for example the logistic map in one dimension and the quadratic map introduced by Hénon [1, 2] in the plane. It is easy to see that any quadratic, one dimensional map with a fixed point is affinely conjugate to the logistic map, $x \mapsto rx(1-x)$. In a similar way, Hénon showed that a generic quadratic area-preserving mapping of the plane can be written in normal form as

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} k + y + x^2 \\ -x \end{pmatrix},$$

which has a single parameter, k.

Hénon's study can be generalized in several directions. Moser [3] studied the class of quadratic, symplectic maps, obtaining a useful decomposition and normal form. For example, when the map is quadratic and symplectic in $\mathbb{R}^{2n}$, Moser [3, 4] showed that it can be written as the composition of two affine symplectic maps and a map of the form

$$\begin{pmatrix} q \\ p \end{pmatrix} \mapsto \begin{pmatrix} q + \nabla W(p) \\ p \end{pmatrix}, \tag{1}$$

where W is a homogeneous cubic polynomial in p. The map given in (1) is a particular example of what we call a quadratic shear.

Definition 1.1. *A quadratic shear is a bijective map of the form*

$$x \mapsto f(x) = x + \frac{1}{2}Q(x), \tag{2}$$

where $Q(x)$ is a vector of homogeneous, quadratic polynomials such that f^{-1} is also a quadratic map.

In this way Moser's result is basically a characterization of all symplectic quadratic shears. One of the remarkable aspects of this is that quadratic symplectic maps necessarily have quadratic inverses. In general we can write a quadratic map on $\mathbb{R}^n$ as the composition of an affine map with a quadratic map that is zero at the origin and is the identity at linear order:

$$x \mapsto f(x) = x_0 + L(x + \frac{1}{2}Q(x)), \tag{3}$$

where $x_0 \in \mathbb{R}^n$, L is a matrix, and $Q(x)$ is a vector of homogeneous, quadratic polynomials. Note that if the map f is volume preserving then it is necessary that L satisfies $\det(L) = 1$. Similarly if f is symplectic, then L must be a symplectic matrix. Of course, the quadratic terms also can not be chosen arbitrarily in these cases.

Polynomial maps are of interest from a mathematical perspective. Much work has been done on the "Cremona maps," that is polynomial maps with constant Jacobians [5]. An interesting mathematical problem concerning such maps is the conjecture proposed by O.T. Keller in 1939:

Conjecture 1.1 (Real Jacobian Conjecture). *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a Cremona map. Then f is bijective and has a polynomial inverse.*

This conjecture is still open. It is known that injective polynomial maps are automatically surjective and have polynomial inverses [6, 7], so it would suffice to prove that f is injective. It is easy to see (cf. lemma [2.1] below) that a quadratic map with constant Jacobian is injective, thus the Jacobian conjecture holds for the quadratic case.

Even if the conjecture is true, the degree of the inverse of a Cremona map could be large. For example, the upper bound for the degree of the inverse of a quadratic map on $\mathbb{R}^n$ is known to be 2^{n-1} [7]. Thus in two dimensions the inverse of a quadratic area-preserving mapping is quadratic, as was discussed by Hénon.

The question of integrability of Cremona maps has been addressed by Moser. In [8], he constructs a family of cubic polynomials that are nonintegrable. This was one of the first attempts to show the possibility of complicated behavior in a simple system, i.e., chaos. Related to this, there is an interesting family of Cremona maps that have exactly one integral, the so called trace maps (cf. [9]). For instance, if we let $\rho(x_1, x_2, x_3) = a + b(x_1 + x_2 + x_3) + c(x_1x_2 + x_1x_3 + x_2x_3) + d(x_1x_2x_3)$ and $T : \mathbb{R}^4 \to \mathbb{R}^4$ be given by

$$T \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} \rho(x_1, x_2, x_3) - x_4 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} ,$$

then T is a cubic Cremona map that has the following integral

$$\begin{aligned} I(x_1, x_2, x_3, x_4) &= x_1^2 + x_2^2 + x_3^2 + x_4^2 - a(x_1 + x_2 + x_3 + x_4) \\ &\quad -b(x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4) \\ &\quad -c(x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4) - d(x_1x_2x_3x_4) \ . \end{aligned}$$

We believe that the class of quadratic maps that have quadratic inverses is an interesting one [4]. The study of such maps hinges upon the characterization of quadratic shears in $\mathbb{R}^n$. For instance, it is known that a necessary and sufficient condition for bijective maps of the form (2) to be quadratic shears is that $DQ(x)Q(x) = 0$. However, simpler characterizations are needed; these are known for the cases $n = 2$ and $n = 3$ [4]. In this paper we extend the results of [4] to higher dimensions, in particular to the case $n = 4$. In addition, we apply Moser's theorem in order to characterize a subfamily of quadratic shears. A simplified proof of his result is provided.

2 Quadratic Shears

It is convenient to rewrite (2) as

$$f(x) = x + \frac{1}{2}M(x)x, \tag{4}$$

where $M : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ is a linear function into the set of $n \times n$ matrices. Since $Q(x) = M(x)x$, M must satisfy the symmetry property $M(x)y = M(y)x$ so that $D_x(M(x)x) = 2M(x)$. Thus

$$Df(x) = I + M(x) \ ,$$

and so there is a unique M for any quadratic Q.

In this section we study characterizations of quadratic shears in $\mathbb{R}^n$. First we show that a necessary and sufficient condition for a map of the form (4) to be a quadratic shear is that $M(x)^2x = 0$. After some work, we will see that the matrix must also satisfy $M(x)^3 = 0$.

In the penultimate section of this paper we will demonstrate that when $n \leq 4$ the matrix M satisfies $M(x)^2 = 0$. Though we do not know if this is true in general, we have been unable to construct an example matrix $M(x)$ such that $M(x)^2 \neq 0$. Whenever $M^2 = 0$, the matrix M has all zero eigenvalues and its largest Jordan blocks are of the form $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, which implies that M has rank at most $\lfloor n/2 \rfloor$.

We begin our characterization of quadratic shears by recalling the following lemma that was obtained in [4].

Lemma 2.1. *Let $f(x) = x + \frac{1}{2}M(x)x$ be a quadratic map of $\mathbb{R}^n$ in standard form. The following statements are equivalent*

1. *For all $x \in \mathbb{R}^n$, $\det(Df(x)) = 1$.*
2. *f is bijective with polynomial inverse.*
3. *$[M(x)]^n = 0$.*

Proof. We will show iii)⇒ii)⇒i)⇒iii).

iii)⇒ii) The condition implies that the matrix $I + M(x)$ is invertible with inverse $I - M(x) + M(x)^2 - \cdots - (-1)^n M(x)^{n-1}$. We can write

$$f(x) - f(y) = \left(I + M\left(\frac{x+y}{2}\right)\right)(x - y).$$

So the function is injective. Using theorem A in [6], we conclude that f is bijective with a polynomial inverse.

ii)⇒i) By assumption, $\det(Df(x))$ and $\det(Df^{-1}(f(x)))$ are polynomials in $x_1, x_2, \ldots, x_n$. However, differentiation of $f^{-1}(f(x)) = x$ gives

$$\det(Df^{-1}(f(x)))\det(Df(x)) = 1 ,$$

and therefore, since both are polynomials, $\det(Df(x))$ has to be a constant independent of x. We notice that $\det(Df(x)) = \det(Df(0)) = \det(I) = 1$.

i)⇒iii) Since $\det(I + M(x)) = 1$ and M is linear in x, then for any $\zeta \neq 0$

$$\det(M(x) - \zeta I) = (-1)^n \zeta^n \det(I + M(-\frac{1}{\zeta}x)) = (-1)^n \zeta^n.$$

This implies that the characteristic polynomial of $M(x)$ is $(-\zeta)^n$ and therefore $[M(x)]^n = 0$. □

At this point, we restrict to the case of quadratic maps in standard form whose inverse is also quadratic, i. e. quadratic shears. The next lemma was obtained by Lomeli and Meiss [4]:

Lemma 2.2. *Let $f(x) = x + \frac{1}{2}M(x)x$ be a quadratic map of $\mathbb{R}^n$. Then f is a quadratic shear if and only if $M(x)^2 x \equiv 0$, for all $x \in \mathbb{R}^n$.*

It is a simple consequence of this lemma and the linearity of M that when f is a quadratic shear, then for all $x, y, z \in \mathbb{R}^n$, the matrix M satisfies the following properties

$$M(M(x)x)M(x)x = 0 . \tag{5}$$

$$M(x)M(y)z + M(y)M(z)x + M(z)M(x)y = 0 . \tag{6}$$

Property (5) implies that for each x, $M(x)x$ is a fixed vector of $f(x)$. Choosing $y = x$ in property (6) gives

$$M(z)M(x)x = M(M(x)x)z = -2M^2(x)z \quad \forall x, z . \tag{7}$$

Therefore

$$M(M(x)x) = -2M^2(x) \quad \forall x . \tag{8}$$

It follows from property (8) that, if for some $x^* \in \mathbb{R}^n$ $M(x^*)x^* = 0$, then $M^2(x^*) = 0$. Also, if $M(x^*)^2 = 0$, then $M(M(x^*)x^*) = 0$. Using this, we obtain the following.

Lemma 2.3. *Let $f(x) = x + \frac{1}{2}M(x)x$ be a quadratic shear. Then for all $x \in \mathbb{R}^n$, $M(x)^3 = 0$.*

Proof. Suppose that, for some $x \in \mathbb{R}^n$, $M(x)x \neq 0$. Then, for any z, property (6) implies

$$M(M(x)x)M(x)z + M(x)M(z)M(x)x + M(z)M(M(x)x)x = 0 \ .$$

From this, the symmetry property and (8) we have

$$M(x)^2 M(x)z + M(x)M(x)^2 z + M(z)M(x)^2 x = 0 \ .$$

Hence, we find that $2M(x)^3 z = -M(z)M(x)^2 x$. But this is zero by lemma [2.2]. □

Example 1. A simple family of quadratic shears is determined by any vector $u \in \mathbb{R}^n$ and a symmetric matrix P such that $Pu = 0$. For all $x, y \in \mathbb{R}^n$ let $M(x)y = (x^T P y)u$. Then

$$M(x)^2 x = (x^T P u)(x^T P x)u = 0 \ .$$

We will see that for the case $n = 3$ every quadratic shear can be expressed in this form.

Example 2. A more general example of a quadratic shear is

$$\frac{1}{2}M(x)x = \frac{1}{2}\sum_{j=1}^{r}(x^T P_j x)u_j \ , \tag{9}$$

where the r matrices P_j are symmetric and for all i, j we have that $P_i u_j = 0$. We will see in the next sections that this is the most general form when $n = 4$. More generally, if M satisfies (9), then since there is a maximum number of independent quadratic forms, we can use a linear coordinate transformation to transform the map to reflect this.

Proposition 2.4. *Choose M as in (9). Then it is always possible to assume that*

$$r \leq n - \left\lceil \frac{\sqrt{9+8n}-3}{2} \right\rceil \ .$$

Proof. Let $k = n - \dim\left(\mathrm{Span}\{u_1, \ldots, u_r\}\right)$.

Since $f(u_j) = u_j$, after a linear change of coordinates, we can assume that the shear is of the form $(q + V(p), p)$ where $V(p)$ is a vector of quadratic forms, $q \in \mathbb{R}^{n-k}$ and $p \in \mathbb{R}^k$. We know that the space of quadratic forms in $\mathbb{R}^k$ has dimension $k(k+1)/2$. If $k(k+1)/2 < (n-k)$ then there are some quadratic forms in V that are linearly dependent, and so with a linear transformation in the $q-$space we can reduce them by one. We can continue doing this, until

$k(k+1)/2 \geq (n-k)$. This implies that $k^2+3k-2n \geq 0$ and therefore $k \geq \lceil \frac{\sqrt{9+8n}-3}{2} \rceil$. Going back to the original function, we let $r = n-k$. $\square$

The following table illustrates the maximum number, r_n, of quadratic forms needed, if the quadratic shear is chosen as in (9). In this case, each quadratic form is a function in at least k_n variables, since $r_n + k_n = n$.

n	1	2	3	4	5	6	7	8	9	10
r_n	0	1	1	2	3	3	4	5	6	6
k_n	1	1	2	2	2	3	3	3	3	4

Table 1: Maximum number r_n of quadratic forms needed. Each of the r_n quadratic forms will be a function of k_n variables.

3 Moser's result and consequences

In this section we use the characterization of quadratic shears in lemma [2.2] to give an alternate proof of the result of Moser [3] for quadratic symplectic maps. As a consequence we are able to characterize quadratic shears for which $M(x)^2 = 0$.

The standard symplectic form, ω, is defined as $\omega(v, v') = v^T J v'$ where J is the $2n \times 2n$ matrix,

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} .$$

A map f is symplectic with respect to ω if $\omega(Dfv, Dfv') = \omega(v, v')$ for all vectors $v, v' \in \mathbb{R}^{2n}$, or consequently when

$$Df^T J Df = J . \tag{10}$$

The main part of Moser's theorem characterizes quadratic symplectic shears.

Theorem 3.1. *Let F be a quadratic symplectic map of $(\mathbb{R}^{2n}, \omega)$. Then F can be decomposed as $F = T \circ S$ where T is affine symplectic and S is a symplectic quadratic shear. Furthermore, if S is any symplectic quadratic shear, then there is a symplectic linear map λ such that $\lambda \circ S \circ \lambda^{-1}(q, p) = (q + \nabla W(p), p)$ where W is a homogeneous cubic polynomial in p.*

Proof. Let $b = F(0)$ and $L = DF(0)$. Clearly L is a symplectic matrix and if we let $T(x) = b + Lx$, then $S = T^{-1} \circ F$ is a symplectic quadratic map. We can write $S(x) = x + \frac{1}{2}M(x)x$, where $M(x)$ is linear in x and satisfies the symmetry property $M(x)y = M(y)x$. By (10), S is symplectic provided

$$(I + M(x))^T J (I + M(x)) = J .$$

Homogeneity of $M(x)$ implies that $M(x)^T J = J^T M(x)$, and $M(x)^T J M(x) = 0$. These conditions imply that

$$M(x)^2 = 0 \; . \tag{11}$$

Lemma [2.2] then implies that S is a quadratic shear.

To finish the proof, we follow Moser [3] and define the null space of M in the following way $\mathcal{N} = \mathcal{N}(M) = \{y \in \mathbb{R}^{2n} : M(y) = 0\}$. Notice that $y \in \mathcal{N}$ if and only if $M(x)y = 0$, for all $x \in \mathbb{R}^{2n}$.

Recall [10] that the $\omega-$orthogonal complement of a subspace $\mathcal{E} \subset \mathbb{R}^{2n}$ is defined by $\mathcal{E}^\perp = \{v \in \mathbb{R}^{2n} : \omega(v, v') = 0, \forall v' \in \mathcal{E}\}$. We will show that $\mathcal{N}^\perp \subset \mathcal{N}$. For that purpose, we will use the following fact: for any $x, y, z \in \mathbb{R}^n$,

$$M(z)M(x)y = M(x-y)^2 z = 0 \; , \tag{12}$$

that follows from lemma [2.2], linearity, symmetry and equations (6) and (11).

Let $u \in \mathcal{N}^\perp$ and $x \in \mathbb{R}^{2n}$. Now for any $y \in \mathbb{R}^{2n}$, (12) implies that $M(x)y \in \mathcal{N}$. Therefore $\omega(y, M(x)u) = y^T J M(x) u = -y^T M(x)^T J u = -\omega(M(x)y, u) = 0$. This implies that $M(x)u = 0$ and hence $u \in \mathcal{N}$. Standard theorems in symplectic geometry (cf. [10]) imply that, in this case, it is possible to find a lagrangian space $\mathcal{F}$ such that $\mathcal{N}^\perp \subset \mathcal{F}^\perp = \mathcal{F} \subset \mathcal{N}$ and a symplectic linear transformation λ such that

$$\lambda(\mathcal{F}) = \{(q,p) \in \mathbb{R}^n \times \mathbb{R}^n : p = 0\}.$$

Clearly, if $S(x) = x + \frac{1}{2}M(x)x$ is a symplectic quadratic shear, then so is $\tilde{S} = \lambda \circ S \circ \lambda^{-1}$. Assume that $\tilde{S}(x) = x + \frac{1}{2}\tilde{M}(x)x$. Then $\lambda(\mathcal{F}) \subset \mathcal{N}(\tilde{M})$. This implies that for all $(q,p) \in \mathbb{R}^n \times \mathbb{R}^n$,

$$\tilde{M}(q,p)(q,p) = \tilde{M}(q,p)(0,p) = \tilde{M}(0,p)(q,p) = \tilde{M}(0,p)(0,p) \; .$$

Since, in general, the matrix $\tilde{M}(0,p)$ can be written in $n \times n$ blocks as

$$\tilde{M}(0,p) = \begin{pmatrix} A(p) & B(p) \\ C(p) & D(p) \end{pmatrix} \; ,$$

then $\tilde{M}(0,p)(q,0) = 0$ implies $A(p) = C(p) = 0$. Moreover, since $\tilde{S}$ is symplectic, we find that $D(p) = 0$ and $B(p)^T = B(p)$. Thus, finally, we see that $\tilde{M}(q,p)(q,p) = (B(p)p, 0)$, where $B(p)p$ is a gradient vector field. □

The following corollary will allow us to simplify a certain class of quadratic shears, as a direct application of Moser's theorem.

Corollary 3.2. *Let $f(x) = x + \frac{1}{2}M(x)x$ be a quadratic shear in $\mathbb{R}^n$. If $M(x)^2 = 0$ then there exists a linear subspace $\mathcal{K} \subset \mathbb{R}^n$ such that*

1. *$\forall x \in \mathcal{K}$, $f(x) = x$ (or $M(x)x = 0$).*

2. *$\forall x \in \mathbb{R}^n$, $M(x)x \in \mathcal{K}$.*

3. *Furthermore, f is linearly conjugate to a map of the form*

$$\begin{pmatrix} q \\ p \end{pmatrix} \mapsto \begin{pmatrix} q + V(p) \\ p \end{pmatrix} ,$$

where V is quadratic in p, $q \in \mathbb{R}^{n-k}$, $p \in \mathbb{R}^k$ and $k \geq \lceil \frac{\sqrt{9+8n}-3}{2} \rceil$.

Proof. It is a well know fact that if $f : \mathbb{R}^n \to \mathbb{R}^n$ is a diffeomorphism, then the following map is symplectic in $\mathbb{R}^n \times \mathbb{R}^n$.

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} f(x) \\ Df(x)^{-T} y \end{pmatrix} .$$

In our case, $Df(x) = I + M(x)$ and, since $M(x)^2 = 0$, $Df(x)^{-T} = I - M(x)^T$. Therefore, if we define $F(x,y) = (x + \frac{1}{2}M(x)x, y - M(x)^T y)$ then F is quadratic and symplectic.

Moser's theorem implies that it is possible to find a homogeneous cubic potential W, and a symplectic matrix λ such that $F = \lambda^{-1} \circ G \circ \lambda$, where $G(x,y) = (x + \nabla W(x), y)$. Assume that the symplectic matrix is

$$\lambda = \begin{pmatrix} A & B \\ C & D \end{pmatrix} .$$

The symplectic condition (10) implies that the inverse of this matrix is

$$\lambda^{-1} = \begin{pmatrix} D^T & -B^T \\ -C^T & A^T \end{pmatrix} ,$$

and since λ^T is also symplectic, then $CD^T = DC^T$. This implies that $F(x,0) = \lambda^{-1} G(Ax, Cx) = \lambda^{-1}(Ax + \nabla W(Cx), Cx) = (x + D^T \nabla W(Cx), -C^T \nabla W(Cx))$. Hence,

$$D^T \nabla W(Cx) = \frac{1}{2} M(x)x ,$$

and

$$C^T \nabla W(Cx) = 0 .$$

Let $\mathcal{K} = \mathrm{Ker}(C)$. Notice that $\mathcal{K} = \{0\}$ implies that $W \equiv 0$, thus we may assume $\mathcal{K} \neq \{0\}$. To finish the proof, it is enough to notice that for all $x \in \mathbb{R}^n$, $\frac{1}{2}M(x)x \in \mathcal{K}$ since

$$CD^T \nabla W(Cx) = DC^T \nabla W(Cx) = 0 .$$

The third part of the corollary follows from the first two, after a linear change of coordinates and proposition [2.4]. $\square$

4 Dimensions Three and Four

Following corollary [3.2], we would like to establish the stronger result that $M(x)^2 = 0$ for all x. In this section we show that this is true when $n = 3, 4$.

Lemma 4.1. *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a quadratic shear where $n \leq 4$. Then, for all x, $M(x)^2 = 0$.*

Proof. Recall that by lemma 2.3, $M(x)^3 = 0$. When $n = 1$, this means that $M(x) = 0$; i.e., the trivial result that there are no quadratic diffeomorphisms in one dimension. When $n = 2$ nilpotency of M implies that $M(x)^2 = 0$ directly. Now consider $n = 3$ or 4. If $M(x)x = 0$ then (8) implies that $M(x)^2 = 0$. Hence assume that there is some x such that $M(x)x \neq 0$. Suppose that for some z, $M(x)^2 z = u \neq 0$. Then the Jordan form of $M(x)$ has one 3×3 block, and for $n = 4$ an additional 1×1 block. $M(x)^2 x = 0$ implies that $M(x)x$ is in $\mathrm{Ker}(M(x))$. Furthermore, $\mathrm{Ker}(M(x)) \cap \mathrm{Range}(M(x)) = \mathrm{Span}\{u\}$. Thus $M(x)x = cu$ for some scalar $c \neq 0$. Thus, from (7), $M(z)M(x)x = -2M(x)^2 z = -2u$. But $M(z)cu = -2u$ is impossible since $M(z)$ is nilpotent. This contradicts $M(x)x \neq 0$. $\square$

Using this lemma for $n = 3$, we can apply corollary 3.2 to directly obtain the following.

Corollary 4.2. *For $n = 3$, for all x, $M(x)x = (x^T P x)u$, where P is symmetric and $Pu = 0$.*

Finally, corollary 3.2 also applies to the case $n = 4$.

Corollary 4.3. *For $n = 4$, for all x there exist vectors u_1 and u_2 and symmetric matrices P_1 and P_2 such that $P_i u_j = 0$ for $i, j = 1, 2$ and $M(x)x = (x^T P_1 x)u_1 + (x^T P_2 x)u_2$.*

5 Conclusion

Any quadratic, volume preserving diffeomorphism that has a quadratic inverse can be written in the form

$$f(x) = a \circ \tau(x)$$

where $a(x) = f(0) + Df(0)x$ is an affine volume preserving map, and $\tau(x) = x + \frac{1}{2}M(x)x$ is a quadratic shear. When $n \leq 4$ we showed that $M(x)^2 = 0$. Though we know of no counter-example to this condition we have only been able to show that $M(x)^3 = 0$ for $n > 4$. When $M(x)^2 = 0$, then there is an additional linear transformation λ such that

$$\tilde{\tau}(x) = \lambda \circ \tau \circ \lambda^{-1}$$

where the quadratic shear $\tilde{\tau}$ takes a particularly simple form

$$\tilde{\tau}\begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} q + V(p) \\ p \end{pmatrix} ,$$

for $(q, p) \in \mathbb{R}^r \times \mathbb{R}^k$, and $V(p)$ a homogeneous quadratic function. We have seen that

$$r \leq n - \left\lceil \frac{\sqrt{9+8n} - 3}{2} \right\rceil$$

In particular when $n = 3$, then $r \leq 1$ and so there is at most a single quadratic function of two variables, and when $n = 4$, $r \leq 2$, so there is either pair of quadratic functions in two variables, or a single quadatic function of three variables.

The dynamics of this class of maps is certainly a least as rich as those of the Hénon map, and we believe their study will prove equally enlightening.

References

[1] M. Hénon. Numerical study of quadratic area-preserving mappings. *Q. J. Appl. Math.*, 27:291–312, 1969.

[2] M. Hénon. A two-dimensional mapping with a strange attractor. *Comm. Math. Phys.*, 50:69–77, 1976.

[3] J. K. Moser. On quadratic symplectic mappings. *Math. Zeitschrift*, 216:417–430, 1994.

[4] H.E. Lomelí and J.D.Meiss. Quadratic volume-preserving maps. *Nonlinearity*, 11:557–574, 1998.

[5] W. Engel. Ganze Cremona-transformationen von Prinzahlgrad in der Ebene. *Math. Ann.*, 136:319–325, 1958.

[6] W. Rudin. Injective polynomial maps are automorphisms. *The American Mathematical Monthly*, 102:540–543, 1995.

[7] H. Bass, E. H. Cornell, and D. Wright. The Jacobian conjecture: Reduction of degree and formal expansion of inverse. *Bull. Amer. Math. Soc.*, 7:287–330, 1982.

[8] J. K. Moser. On the integrability of area preserving cremona mappings near an elliptic fixed point. *Bol. Soc. Mat. Mexicana (2)*, 5:176–180, 1960.

[9] J Roberts and M. Baake. Trace maps as 3D reversible dynamical systems with an invariant. *Journal of statistical physics*, 74(3):829–888, 1994.

[10] R. Abraham and E. Marsden. *Foundations of Mechanics*. Benjamin Cummings, 1985.