The Last Invariant Torus

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Symplectic Twist Maps

Poincaré Sections of Natural Hamiltonians

$$\frac{dz}{dt} = J\nabla H(z) \quad z = (q,p) \qquad J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

$$H(q,p) = T(p) + V(q)$$

Symplectic Twist Maps

Froeshlé Map

$$\begin{aligned} x' &= x + Ty' \mod 1 \\ y' &= y - \nabla V(x) \end{aligned}$$

Symplectic if T symmetric $\quad f^*\omega = \omega$

$$\omega = dx \wedge dy = \sum_{i=1}^{n} dx_i \wedge dy_i$$

Exact Symplectic (zero net flux)

$$f^*(ydx) - ydx = dL$$

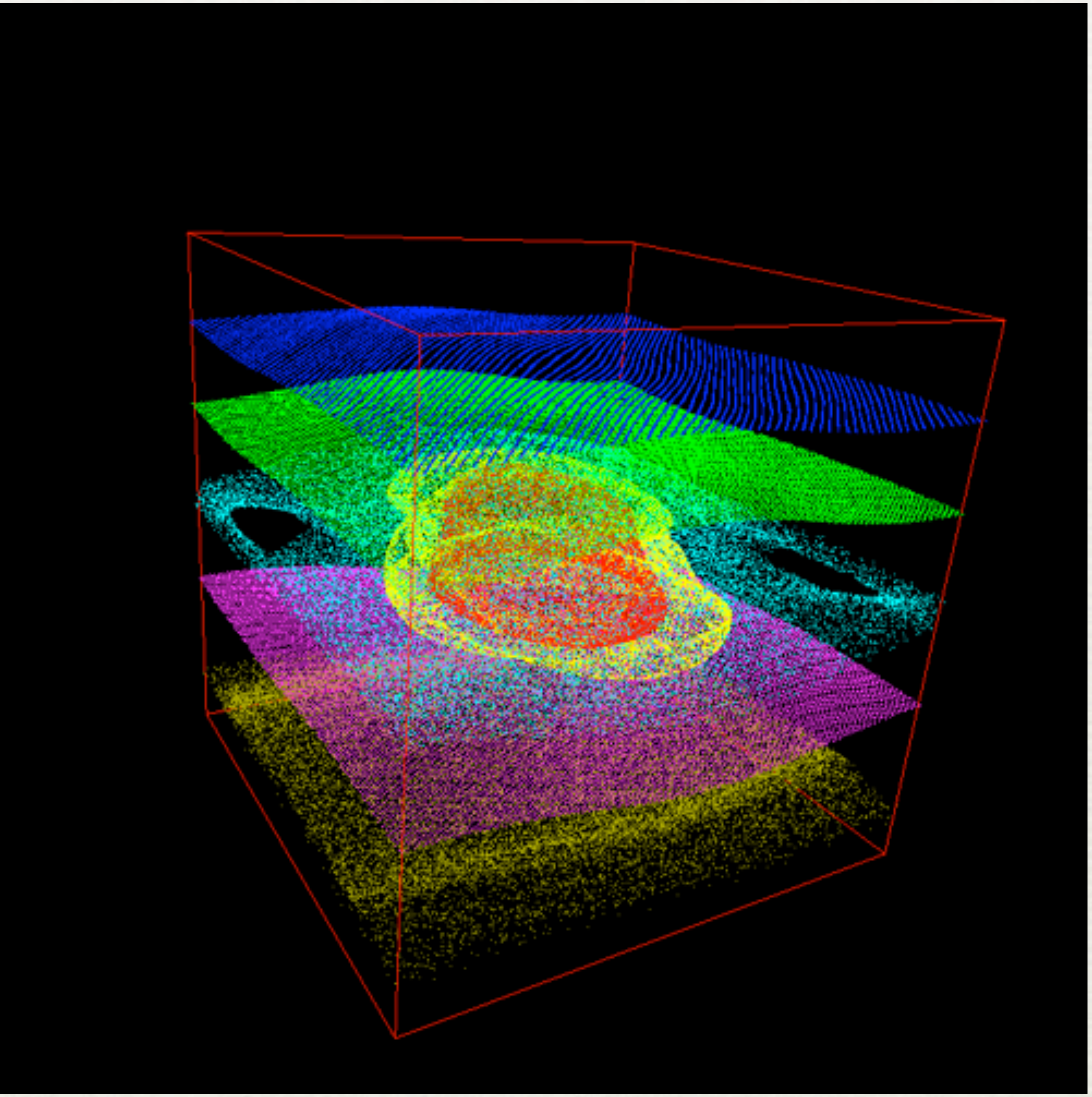
Twist Condition: $\quad T = K^{-1} > 0$

$$L(x, x') = \tfrac{1}{2}(x' - x)^T K(x' - x) - V(x)$$

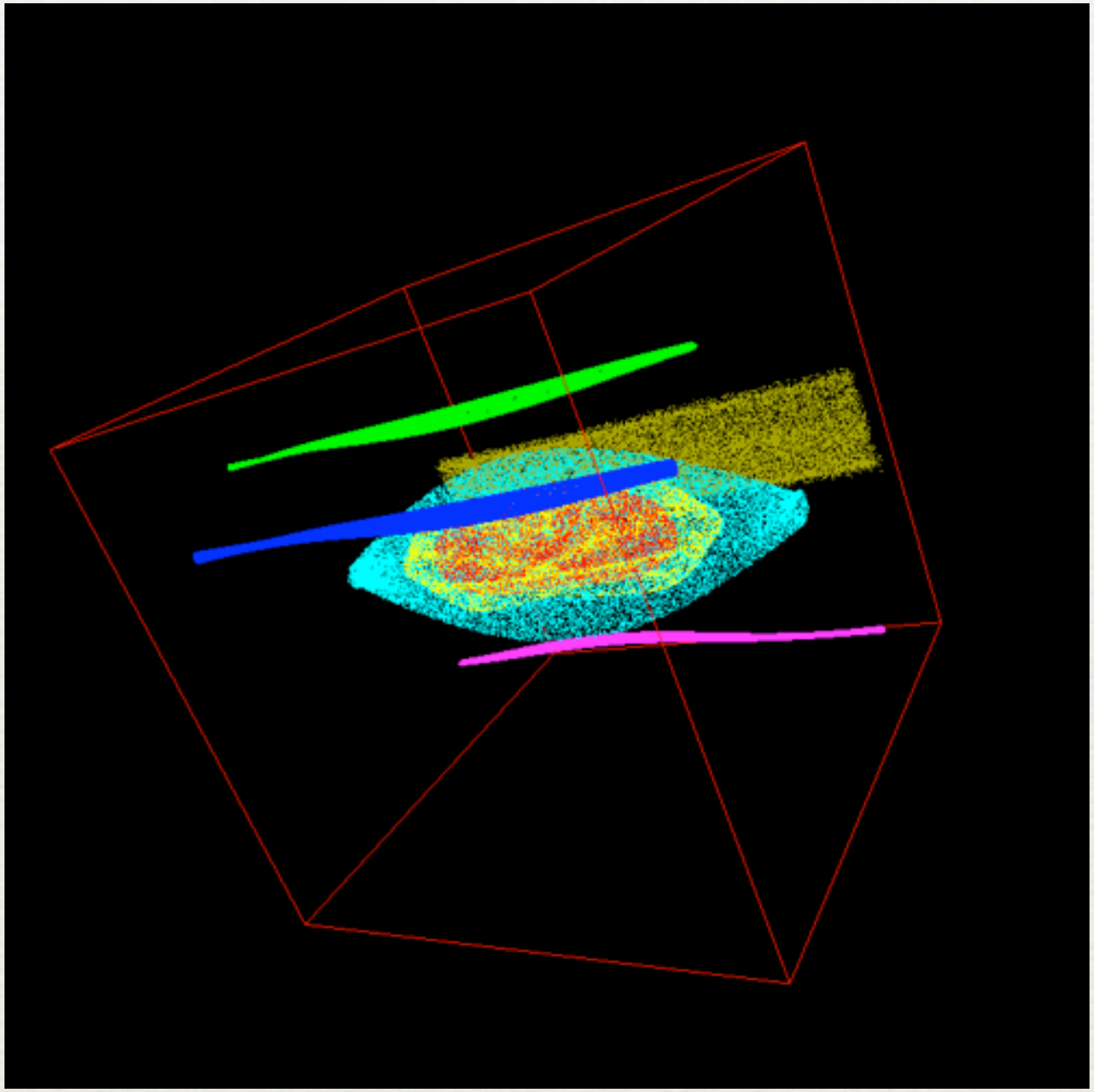
$$\begin{aligned} y &= -L_1 = K(x' - x) \\ y' &= L_2 = K(x' - x) - \nabla V(x) \end{aligned}$$

$$T = I$$

$$V(x) = \frac{1}{4\pi^2}\left(a\cos(2\pi x_1) + b\cos(2\pi x_2) + c\cos(2\pi(x_1 + x_2))\right)$$



y_2 projection



x_1 projection

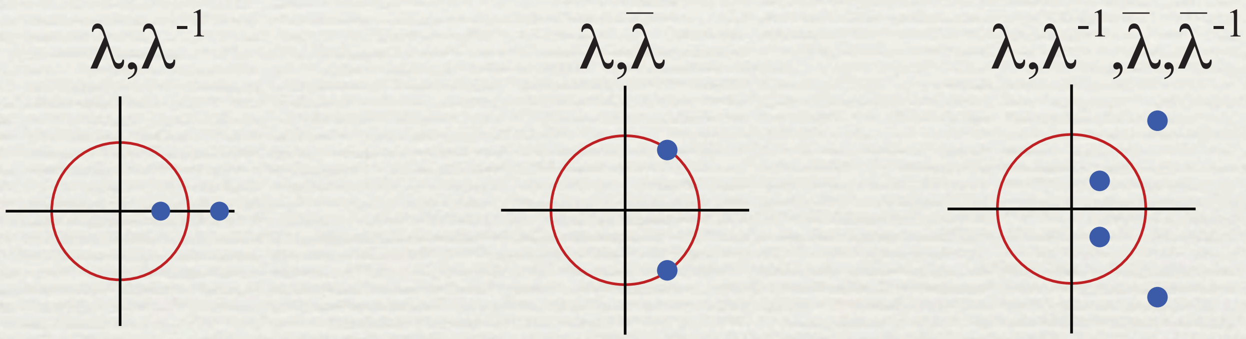
a=0.1, b=0.2 c = 0.1

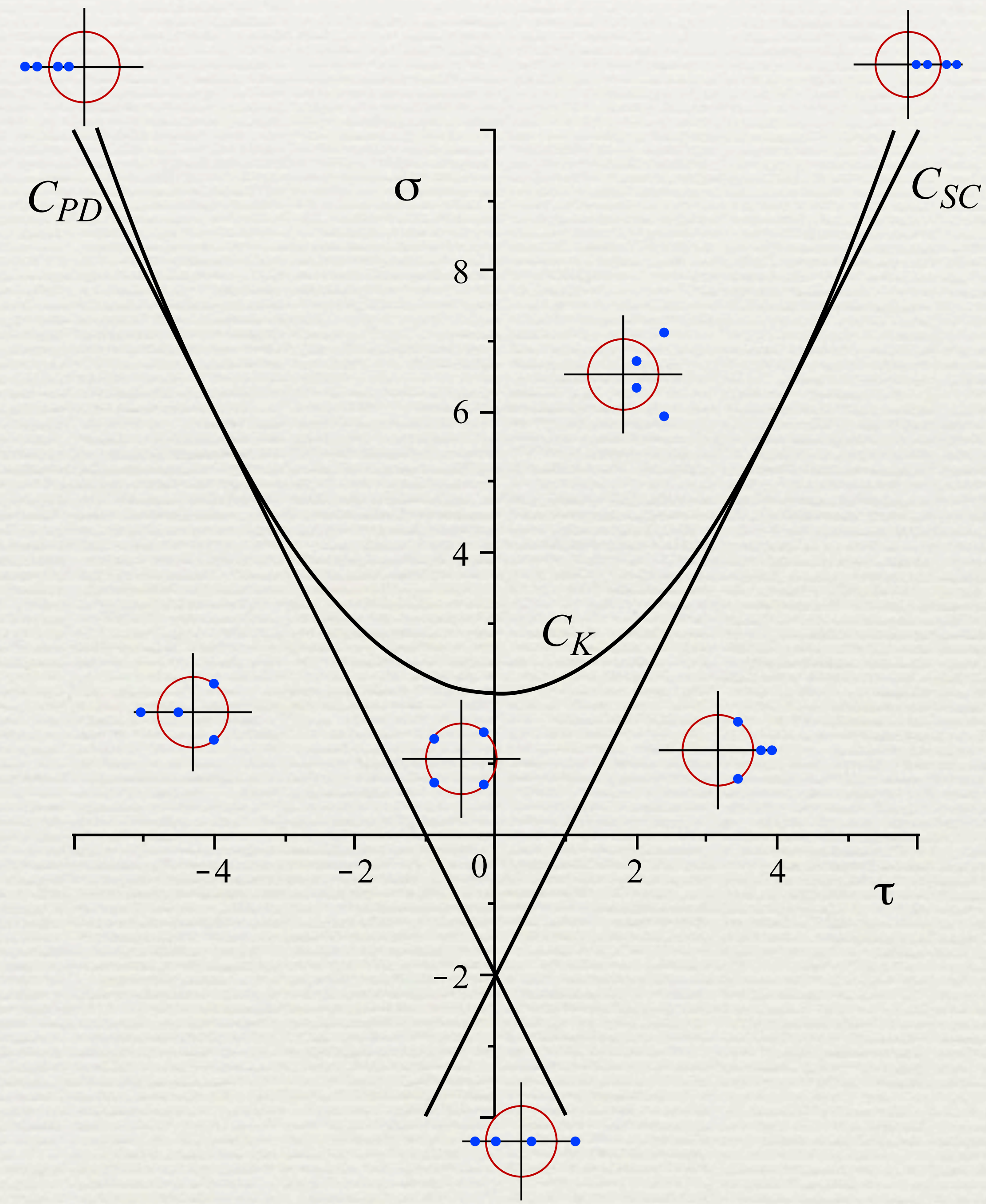
Symplectic Stability

Characteristic Polynomial is reflexive

$$\det(Df - \lambda I) = \lambda^4 - \tau\lambda^3 + \sigma\lambda^2 - \tau\lambda + 1$$

Pairs & Quartets of Multipliers





Frequency Maps

Frequency: Action to ω $\Omega : \mathbb{R}^n \to \mathbb{R}^n$

Froeshlé case
$\Omega(y) = Ty$

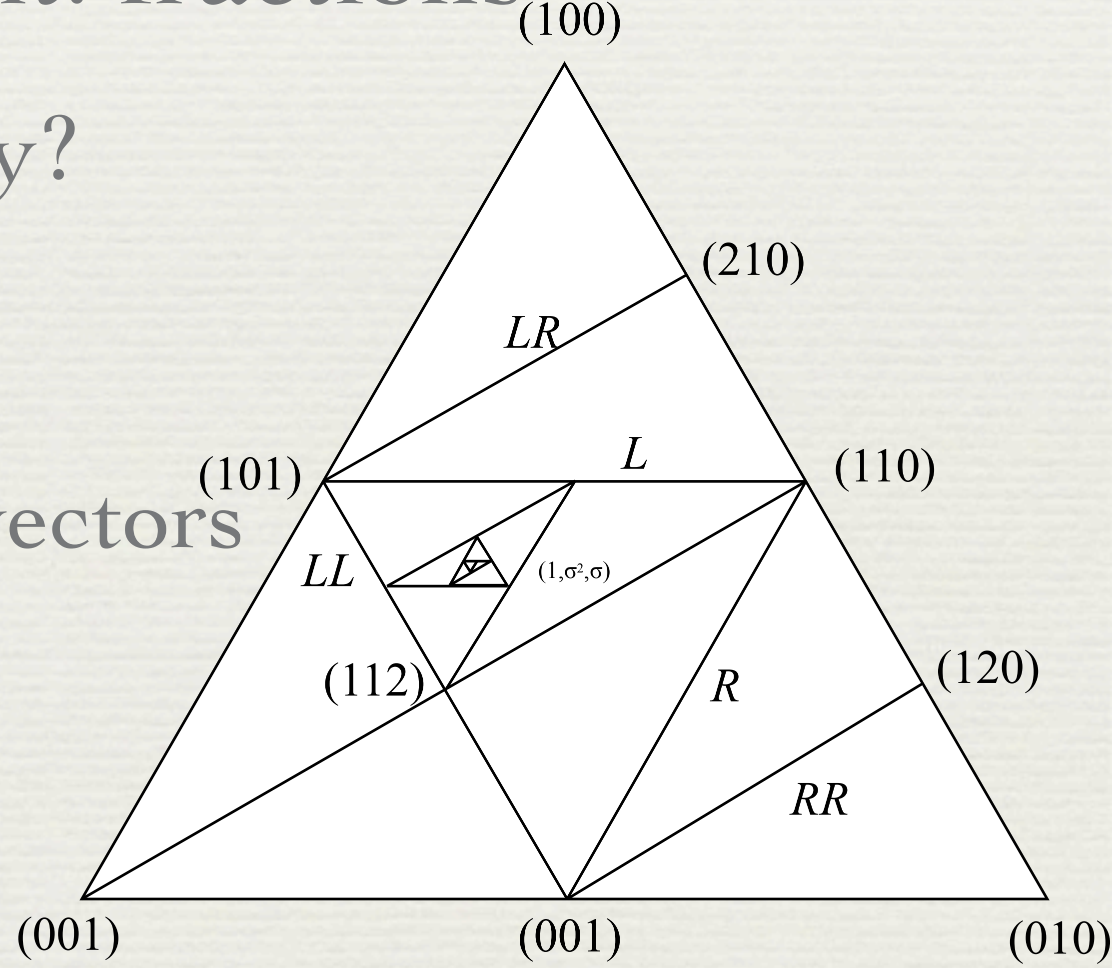
KAM: assumes Ω is a local diffeomorphism

No good theory of multi-dim cont. fractions

- Best approximates? Periodicity?

Farey Tree generalization

- binary encoding of frequency vectors
- natural self-similar patterns
- spiral mean $\sigma^3 = \sigma + 1$
- Diophantine vector (σ, σ²,1)



Kim, S. and S. Ostlund (1986). "Simultaneous Rational Approximations in the Study of Dynamical Systems." Phys. Rev. A **34**: 3426-3434.

A Tale of Three Methods

Converse KAM Theory
Frequency Analysis
Crossing Time

#1 Converse KAM Theory

Nonexistence criteria

- Birkhoff's Theorem: every rotational invariant circle of an area-preserving twist map is a graph $\{(x, S'(x)) : x \in \mathbb{T}^2\}$
  - Confinement Corollary: if all orbits below $y = a$ remain below $y = b$, then there is a rotational invariant circle in (a,b).
  - By twist condition, S is Lipschitz

Cone Criterion

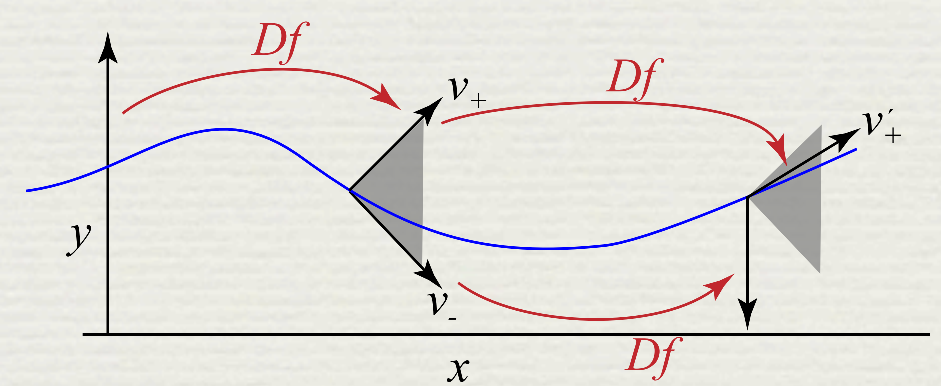
$$Df = \begin{pmatrix} 1 - F'(x) & 1 \\ -F'(x) & 1 \end{pmatrix}$$

$$v_+ = Df \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$v_- = Df^{-1} \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 + F' \end{pmatrix}$$

$$v'_+ = Df v_+ = \begin{pmatrix} 2 - F' \\ 1 - F' \end{pmatrix}$$

v'_+ below cone if $|F'(x)| > \sqrt{3}$



Converse KAM Theory

Mather: no rotational circles if $k > 4/3$

- Rigorous numerics: if $k > 63/64$

Mather, J. N. (1984). "Non-Existence of Invariant Circles." Ergodic Theory and Dynamical Systems **4**: 301-309.
MacKay, R. S. and I. C. Percival (1985). "Converse KAM: Theory and Practice." Comm. Math. Phys. **98**: 469-512.

Higher Dimensions?

- Analogue of Birkhoff's theorem: Lagrangian invariant tori on which dynamics is chain recurrent are graphs

$$\{(x, \nabla S(x)) : x \in \mathbb{T}^2\}$$

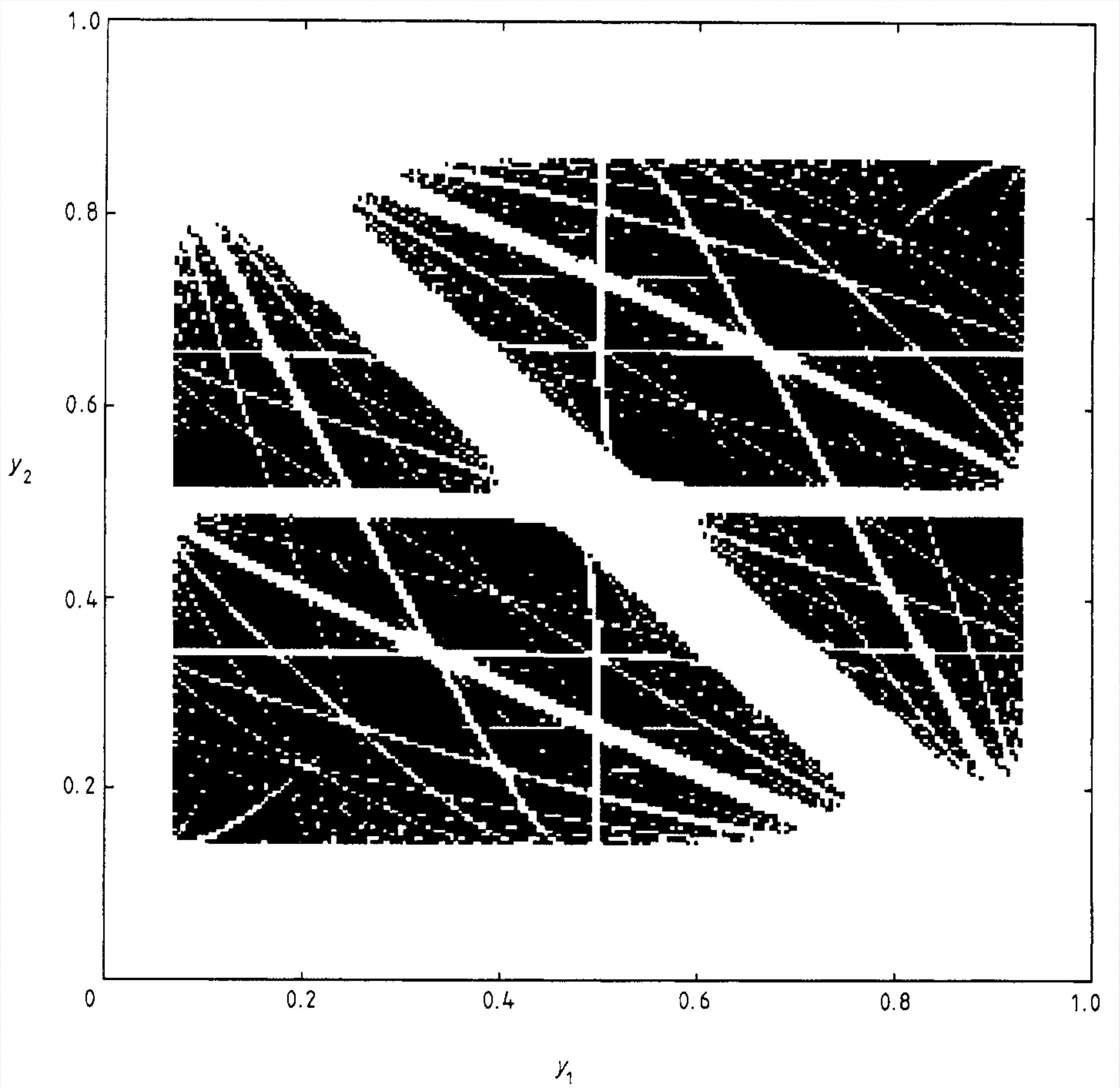
Bialy, M. L. and L. Polterovich (1992). "Hamiltonian Systems, Lagrangian Tori and Birkhoff's Theorem." Math. Ann. **292**: 619-627.

- Every orbit on an invariant Lagrangian graph as minimal action

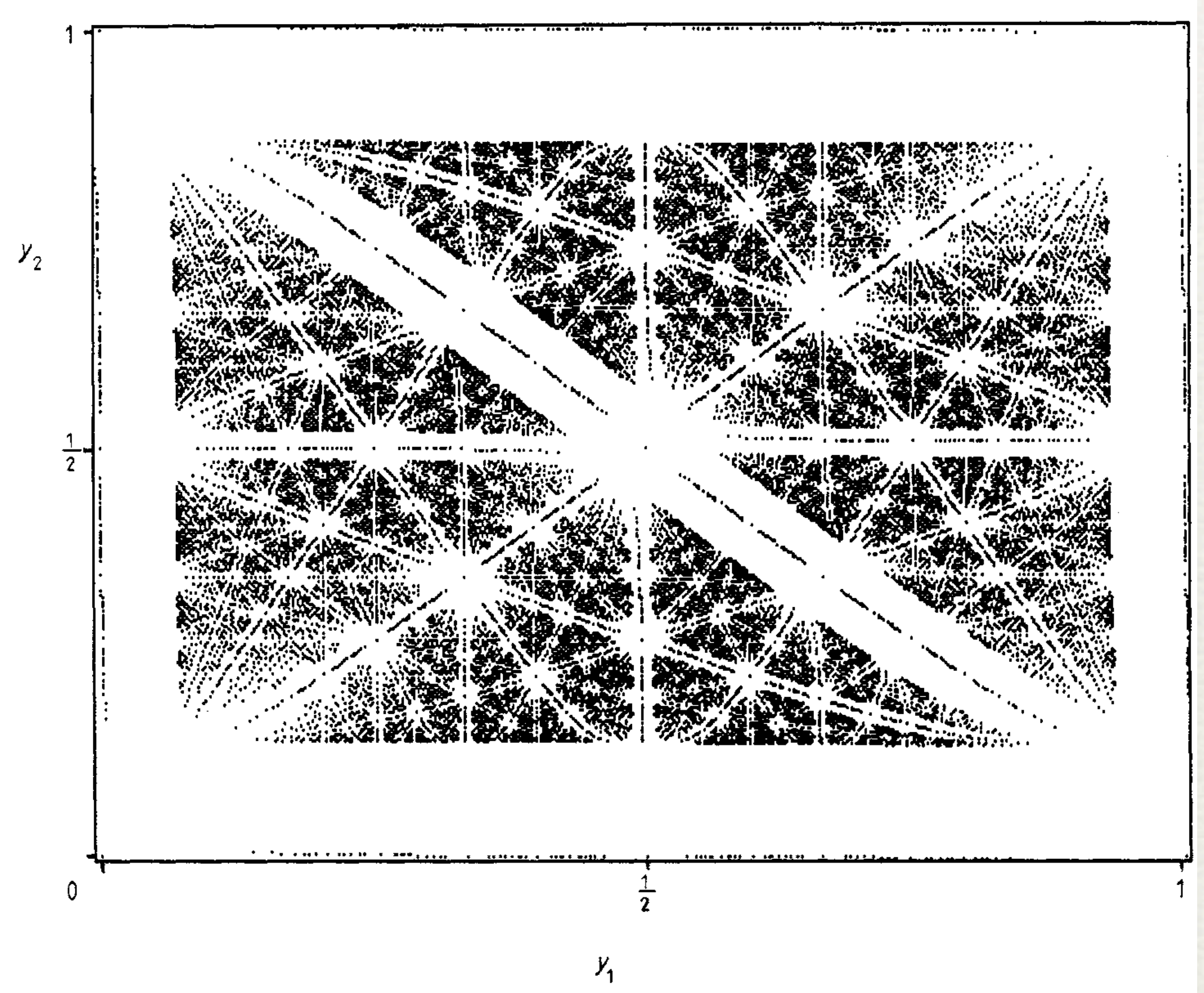
- Simplest criterion: need $|a| + |c| < 2$ and $|b| + |c| < 2$ for tori.

Higher Dimensions

Froeshlé Map, a = 0.05, b = 0.2, c = 0.02



symmetry plane x = 0



symmetric periodic orbits

MacKay, R. S., J. D. Meiss and J. Stark (1989). "Converse KAM Theory for Symplectic Twist Maps." Nonlinearity **2**: 555-570.

#2 Frequency Analysis

KAM Tori have well-defined rotation vectors

$$\omega = \lim_{t\to\infty} \frac{x_t - x_0}{t}$$

Each torus crosses every plane x = *const*, so sufficient to look on one

- for example, symmetry plane Fix(S_1) ={x = 0}.

Finite time approximate frequencies

May use windowed-FFT methods (Laskar)

Frequency Analysis

Birkhoff: invariant circles are graphs

Twist $\Rightarrow$ frequency is monotone on circles

Irregular Frequency Maps $\Rightarrow$ no tori

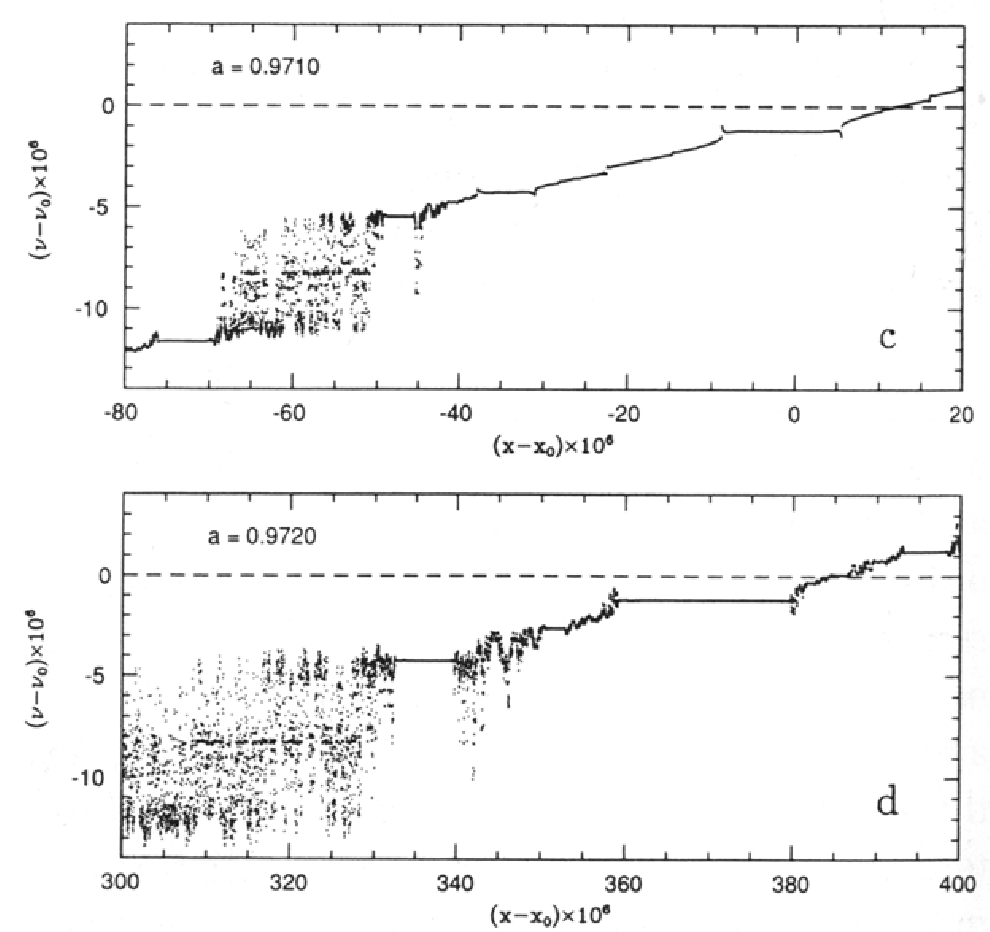


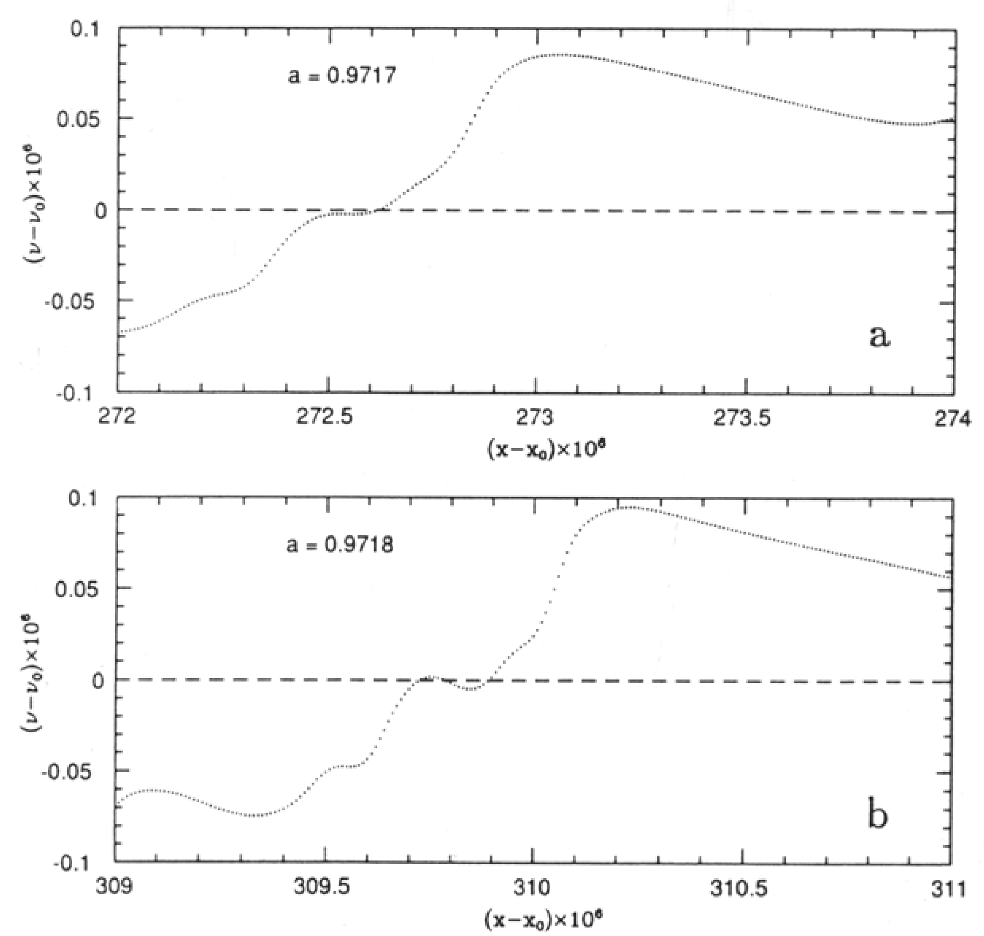
Fig. 2(a)–(d). Variation of the fundamental frequency ν for the standard mapping (13) for different values of the parameter a, in the vicinity of the golden rotation number ν_0 which corresponds to the zero dotted line. The origin in the x scale is arbitrarily taken to be $x_0 = 4.17655$. The origin of frequencies is the golden value $\nu_g = \frac{1}{3}(3-\sqrt{5})$. The unit for ν and x is 10^{-6}. If $x_1 < x_2$ and $\nu(x_1) > \nu(x_2)$, we can conclude that there exist no KAM invariant curves of irrational rotation number between $\nu(x_2)$ and $\nu(x_1)$. In fig. 9b, we can see that the golden invariant curve does not persist for $a = 0.9718$.

Laskar, J. (1993). "Frequency Analysis for Multi-Dimensional Systems. Global Dynamics and Diffusion." Physica D 67: 257-283.

Frequency Analysis

Birkhoff: invariant circles are graphs

Twist $\Rightarrow$ frequency is monotone on circles



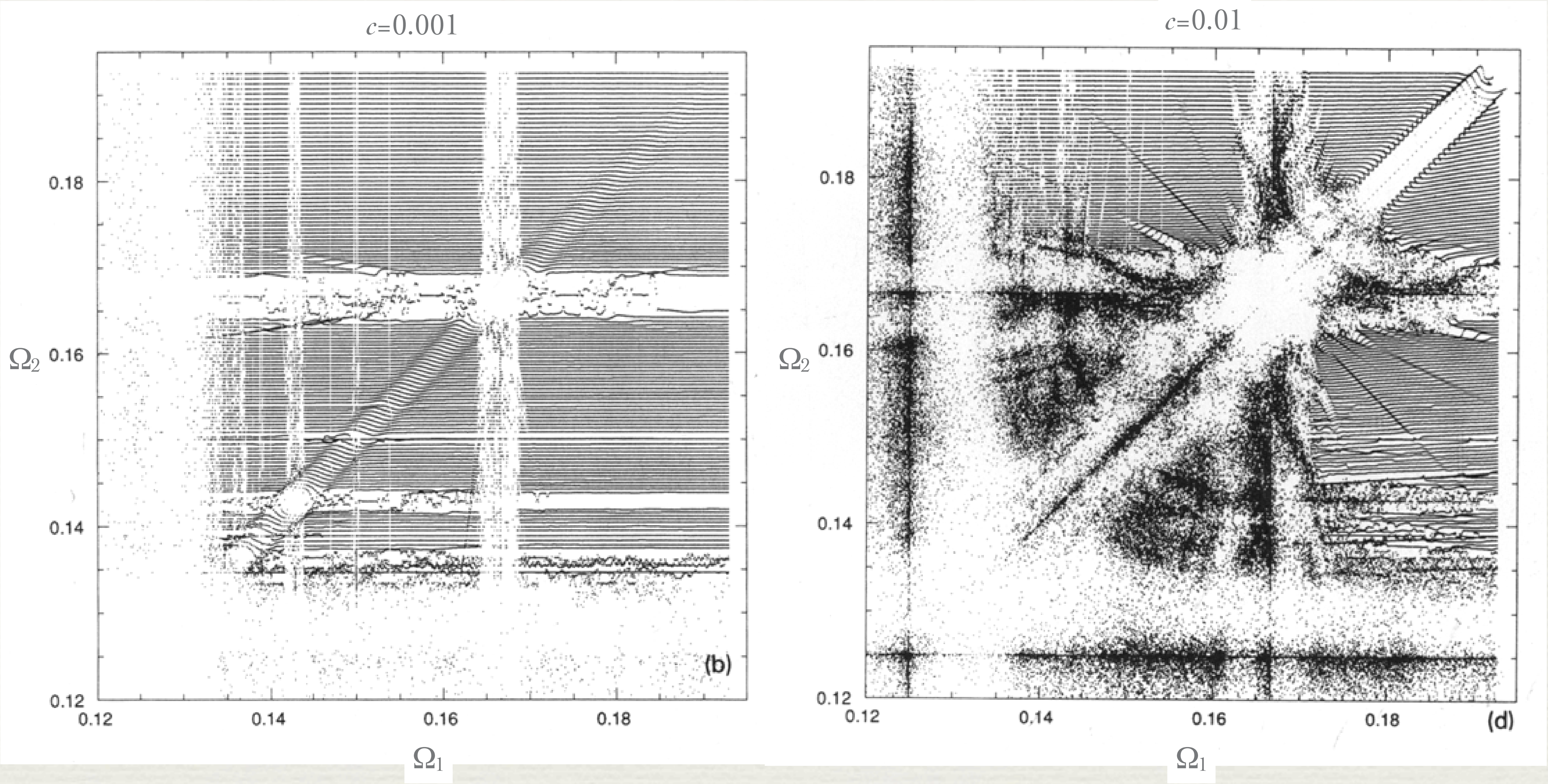
Nonmonotone Frequency Maps $\Rightarrow$ no tori

Fig. 2(a)–(d). Variation of the fundamental frequency ν for the standard mapping (13) for different values of the parameter a, in the vicinity of the golden rotation number ν_0 which corresponds to the zero dotted line. The origin in the x scale is arbitrarily taken to be $x_0 = 4.17655$. The origin of frequencies is the golden value $\nu_g = \frac{1}{3}(3-\sqrt{5})$. The unit for ν and x is 10^{-6}. If $x_1 < x_2$ and $\nu(x_1) > \nu(x_2)$, we can conclude that there exist no KAM invariant curves of irrational rotation number between $\nu(x_2)$ and $\nu(x_1)$. In fig. 9b, we can see that the golden invariant curve does not persist for $a = 0.9718$.

Laskar, J. (1993). "Frequency Analysis for Multi-Dimensional Systems. Global Dynamics and Diffusion." Physica D 67: 257-283.

Higher Dimensions?

Froeshlé Map, $a = b = 1.3$



Laskar, J. (1993). "Frequency Analysis for Multi-Dimensional Systems." Physica D **67**: 257-283.
Dullin, H. R. and J. D. Meiss (2003). "Twist Singularities for Symplectic Maps." Chaos **13**: 1-16.

#3 Crossing Time

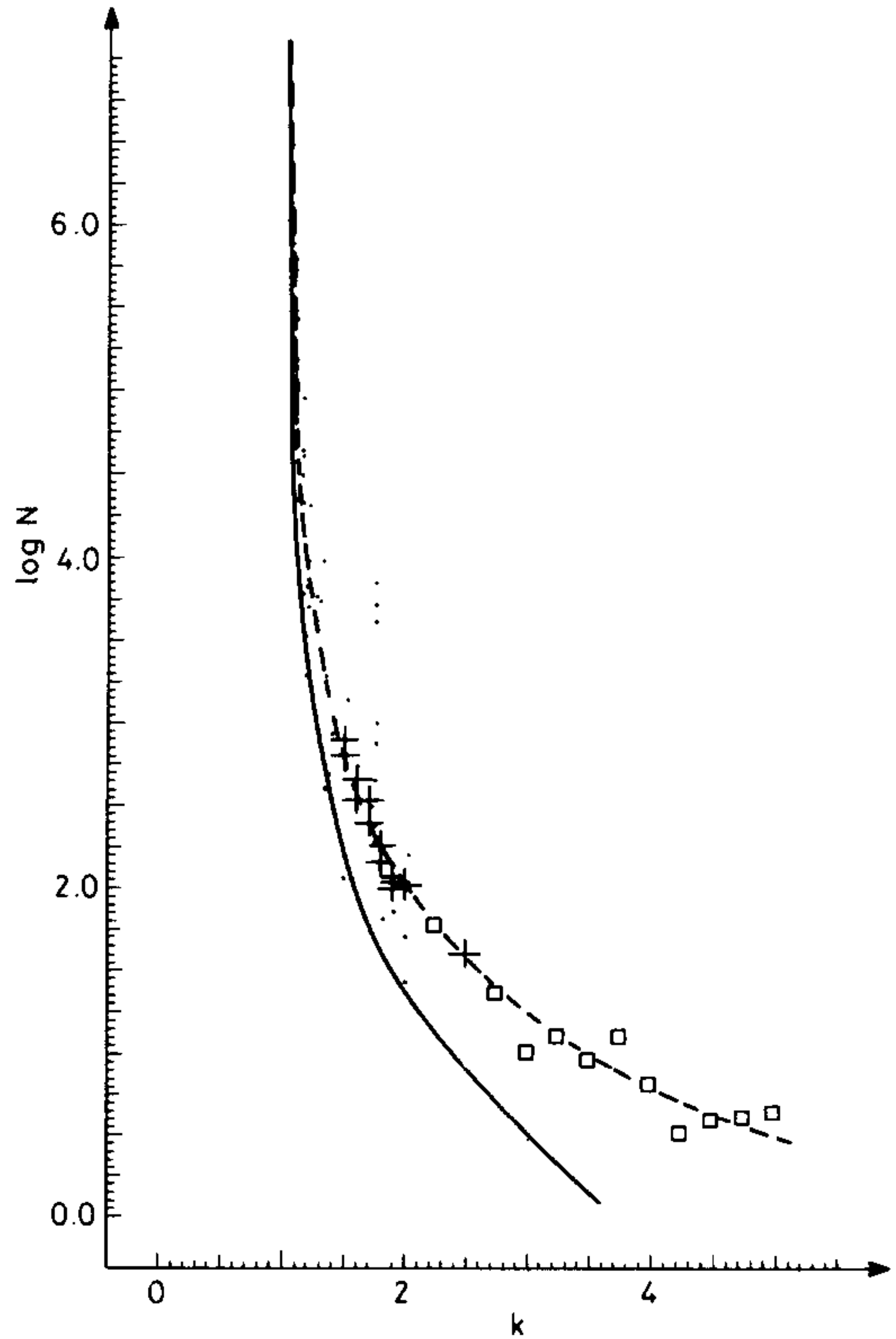


Fig. 16. Comparison of Chirikov's numerical experiments and our formulae. N is the number of iterations to cross the region near the golden cantorus and k is the parameter. The continuous line is for our scaling law (11.7) fitted to the value for the golden cantorus alone, giving $N = 25(\Delta k)^{-\eta}$.

$$N = \frac{C}{(k - k_c)^{\mu}}$$

Chirikov: fit to data

$k_c = 0.989$, $\mu = 2.55$

MMP : cantorus flux

$k_c = 0.971635$,

$\mu = \eta = 3.01177$

$C = 25$

Chirikov, B. V. (1979). "A Universal Instability of Many-Dimensional Oscillator Systems." Phys. Rep. **52**: 265-379.

MacKay, R. S., J. D. Meiss and I. C. Percival (1984). "Transport in Hamiltonian Systems." Physica D **13**: 55-81.

Higher Dimensions?

?

Volume Preserving Maps

Magnetic Field line flows

$$\frac{dx}{dt} = B(x,t) \quad \nabla \cdot B = 0$$

Incompressible Fluids

$$\frac{dx}{dt} = v(x,t) \quad \nabla \cdot v = 0$$

Poincaré Map for Periodic Time dependence: V.P.

Invariant Tori

KAM theory applies to one-action, n-angle, exact V.P. Maps $(x, z) \in \mathbb{T}^n \times \mathbb{R}$

Cheng, C.-Q. and Y.-S. Sun (1990). "Existence of Invariant Tori in Three Dimensional Measure-Preserving Mappings." Celestial Mech. 47(3): 275-292.

$$\begin{aligned} x' &= x + \Omega(z') + \varepsilon g_1(x, z) \quad \mod 1 \\ z' &= z + \varepsilon g_2(x, z) \end{aligned}$$

$$\int_{\mathbb{T}^n} g_2 dx = 0$$

providing $\det(D\Omega, D^2\Omega) \neq 0$.

Cantor sets of invariant tori for $|\varepsilon| \ll 1$, though cannot be identified by fixed frequency vector

$$\omega \in \mathbb{R}^n$$

A Residue Criterion?

KAM theory applies (Cheng & Sun)

- However, can't fix the frequencies!

Is there a last torus? Self-similarity?

- What rotation vector plays the role of the golden mean? Perhaps a cubic irrational: spiral mean $\sigma^3 = \sigma + 1$?

Are there cantori?

- Some results: Anti-integrability by Li & Malkin

Is there an algebraic singularity in the crossing time?

Standard VP Map

3D, one-action map

$$\begin{aligned} x_1' &= x_1 + \Omega_1(z') \ , \\ x_2' &= x_2 + \Omega_2(z') \ , \\ z' &= z + \varepsilon g(x) \ , \end{aligned}$$

For "twist condition" need nonzero curvature

$$\Omega(z) = (z + \gamma, -\delta + \beta z^2) \ .$$

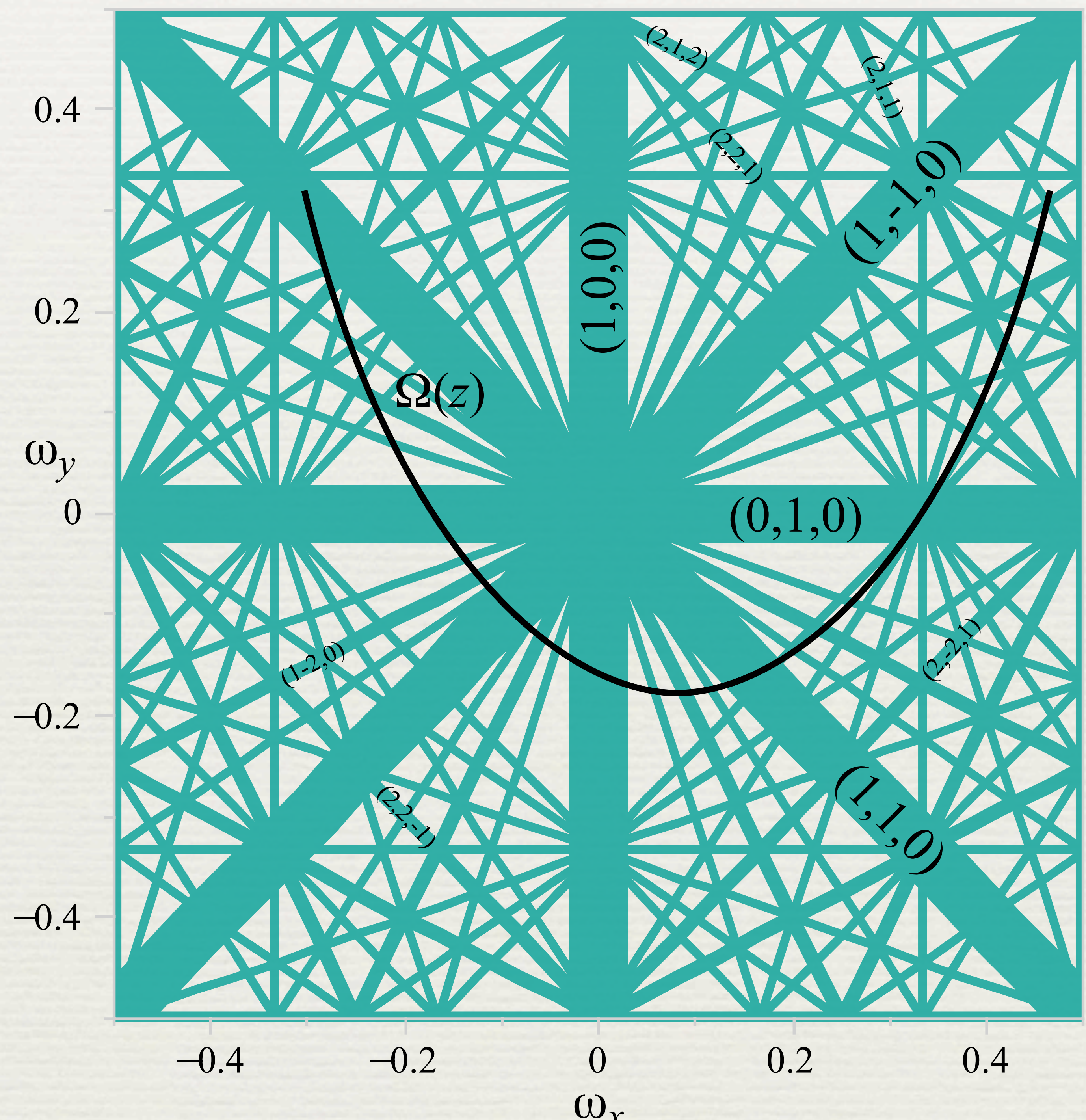
Dullin, H. R. and J. D. Meiss (2010). "Resonances and Twist in Volume-Preserving Mappings." Disc. Cont. Dyn. Sys. submitted.

Forcing any generic periodic function

$$g(x) = -a\sin(2\pi x_1) - b\sin(2\pi x_2) - c\sin(2\pi(x_1 - x_2))$$

$\Omega : \mathbb{R} \to \mathbb{R}^2$

$$\Omega(z) = (z + \gamma, -\delta + \beta z^2)\ .$$



Resonances

$$m \cdot \omega = n$$

Diophantine Condition

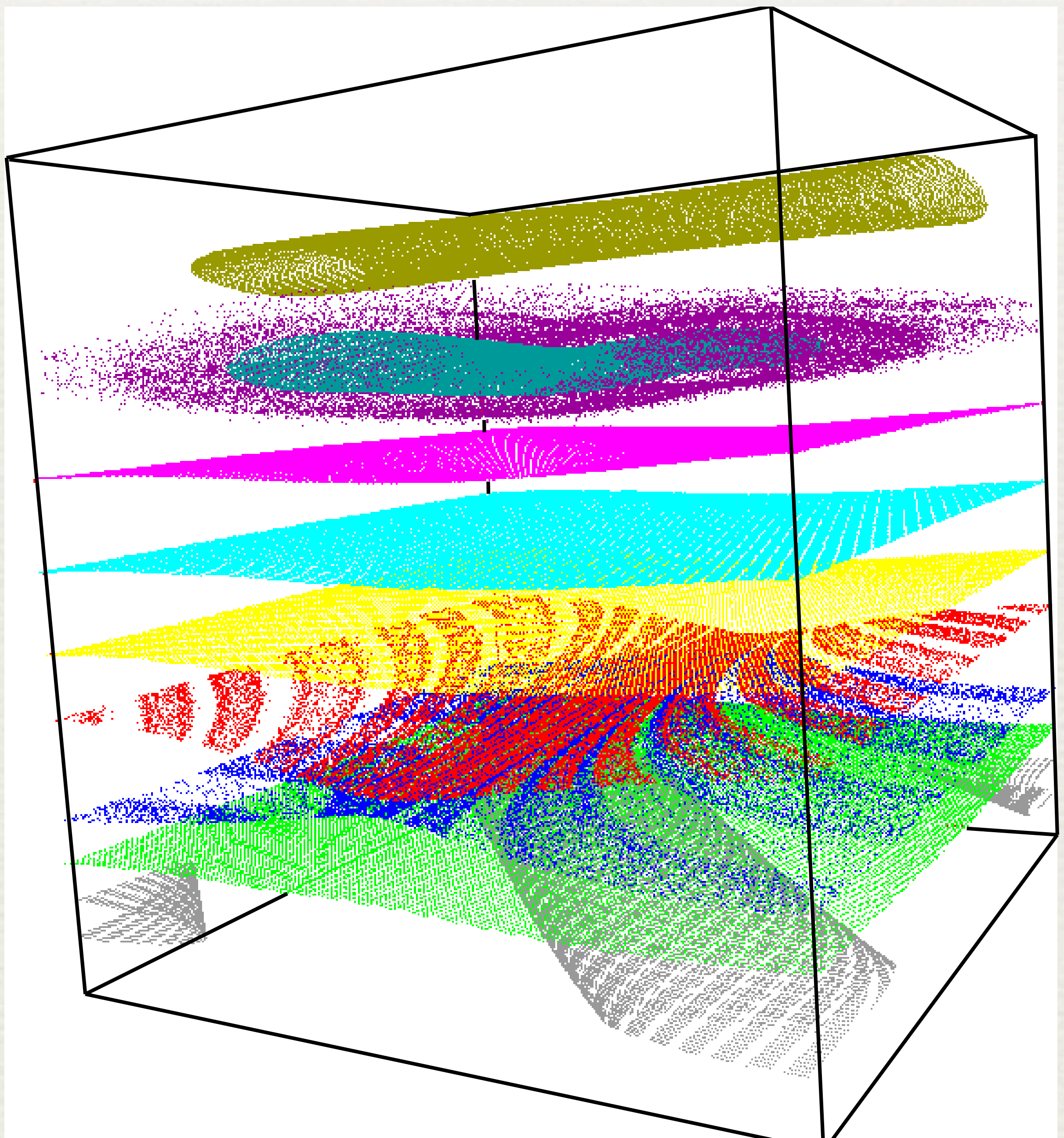
$$|m \cdot \omega - n| > \frac{C}{|m|^\tau}$$

$$\mathcal{R} \equiv \{\omega \in \mathbb{R}^d : m \cdot \omega = n \text{ for some } (m,n) \in \mathbb{Z}^{d+1} \setminus \{0\}\}$$

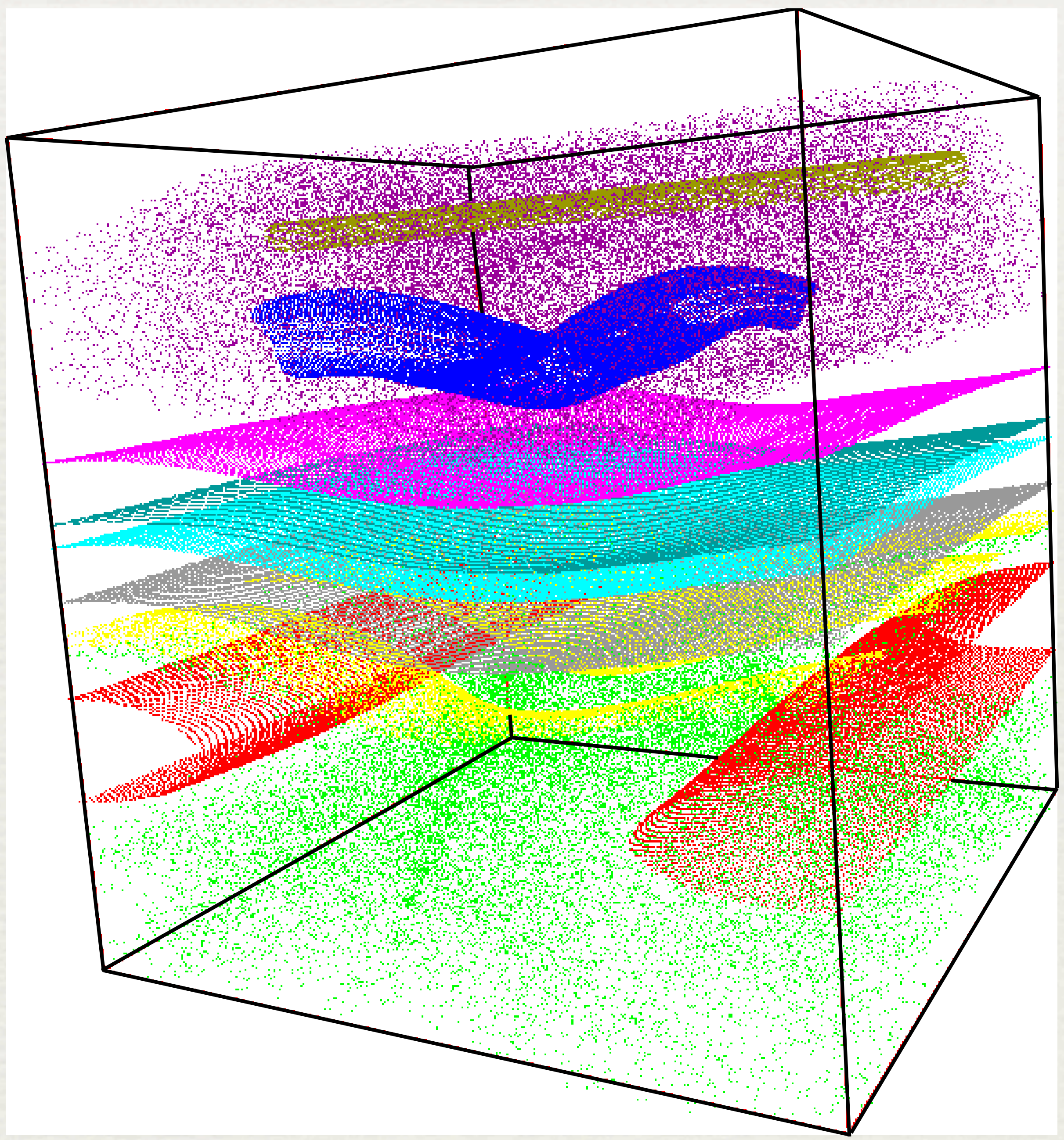
Resonances

Driven (1,0,0), (0,1,0) and (1,1,0) resonances

$\gamma = \frac{1}{2}(\sqrt{5} - 1) \approx 0.61803 \quad \beta = 2 \quad a = b = c = 1.0 \qquad \delta = 0.1$



$\varepsilon = 0.005$

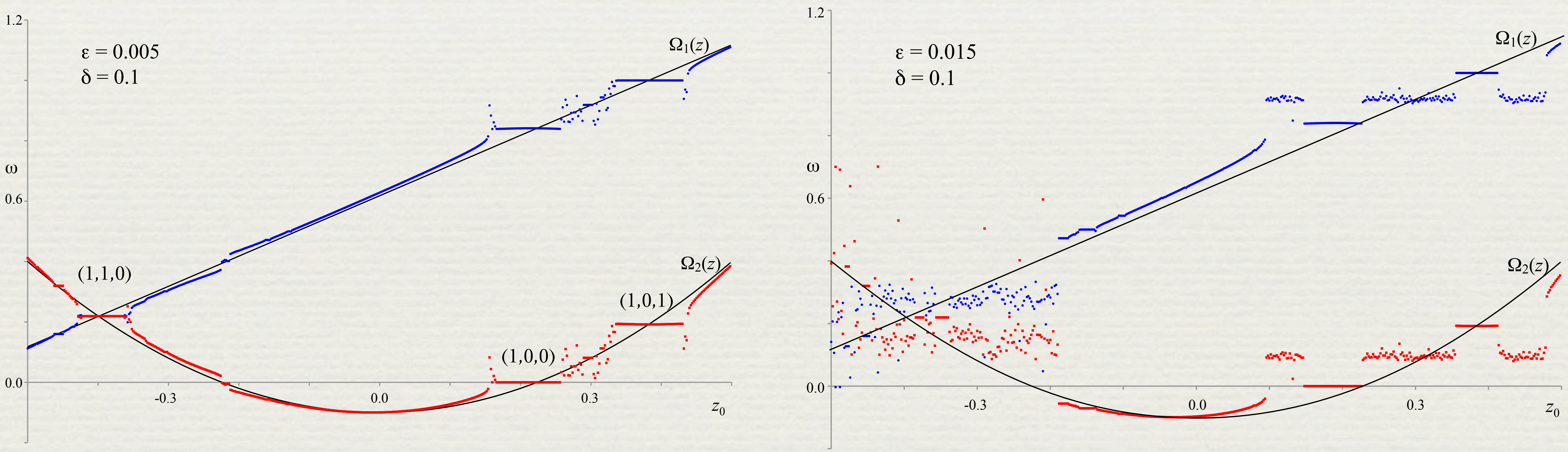


$\varepsilon = 0.015$

Frequency Analysis

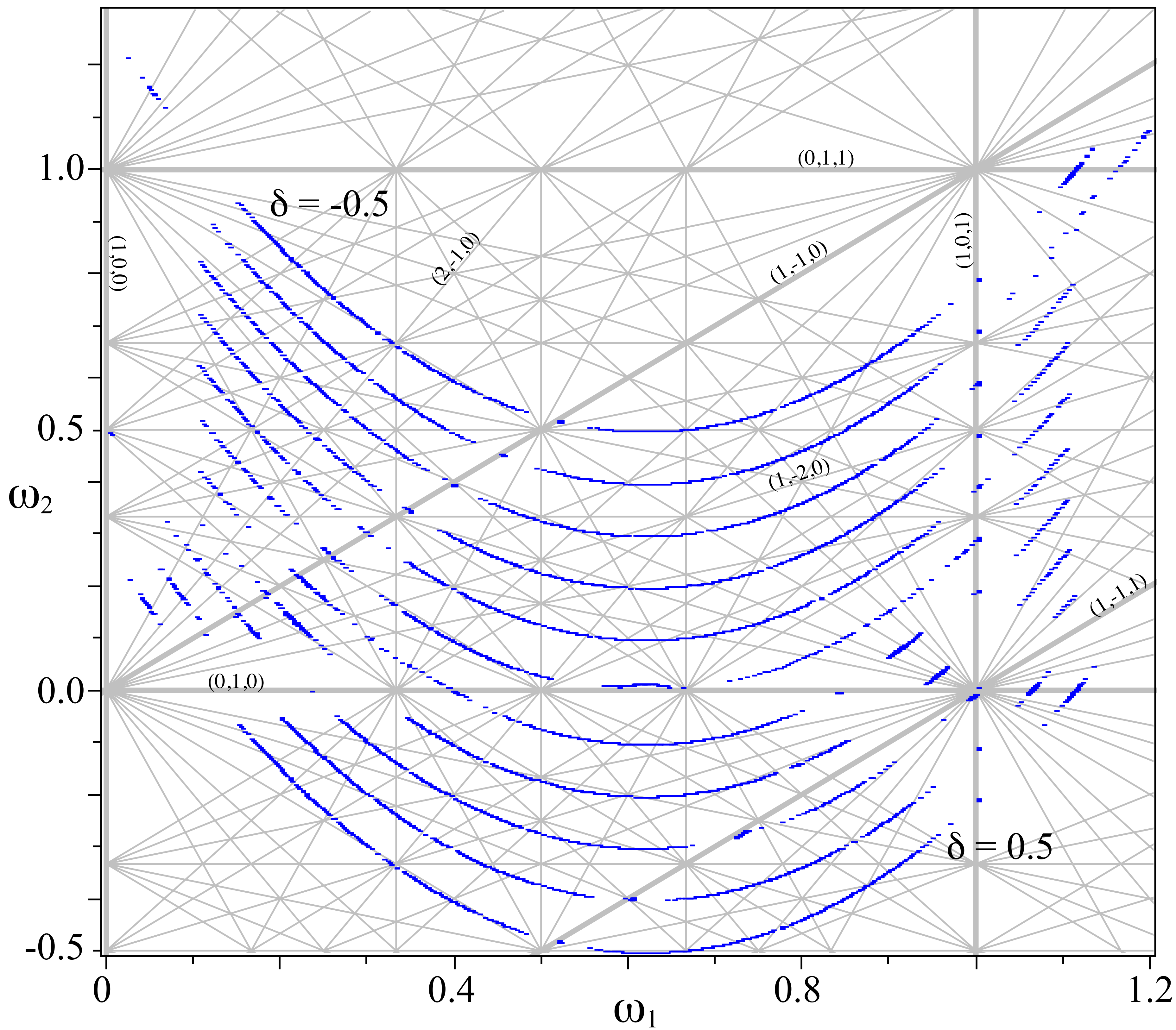
Simplest numerical estimate

$$\omega_T(x_0, z_0) = \frac{x_T - x_0}{T}$$



x_0 = (0,0) $\quad$ T = 10^5

Frequency Maps $\varepsilon = 0.01$



The Last Torus

Easiest case: z-periodic structure

$$\Omega(z+1) = \Omega(z)+m$$

Any invariant set for $z \in [0,1]$ repeated in $[k,k+1]$.

To test for invariant tori need to bound vertical extent

$$\Delta(\mathcal{C}) = max_{\mathcal{C}}(z) - min_{\mathcal{C}}(z)$$

Experiments indicate Δ_{max} is small, say < 0.1.

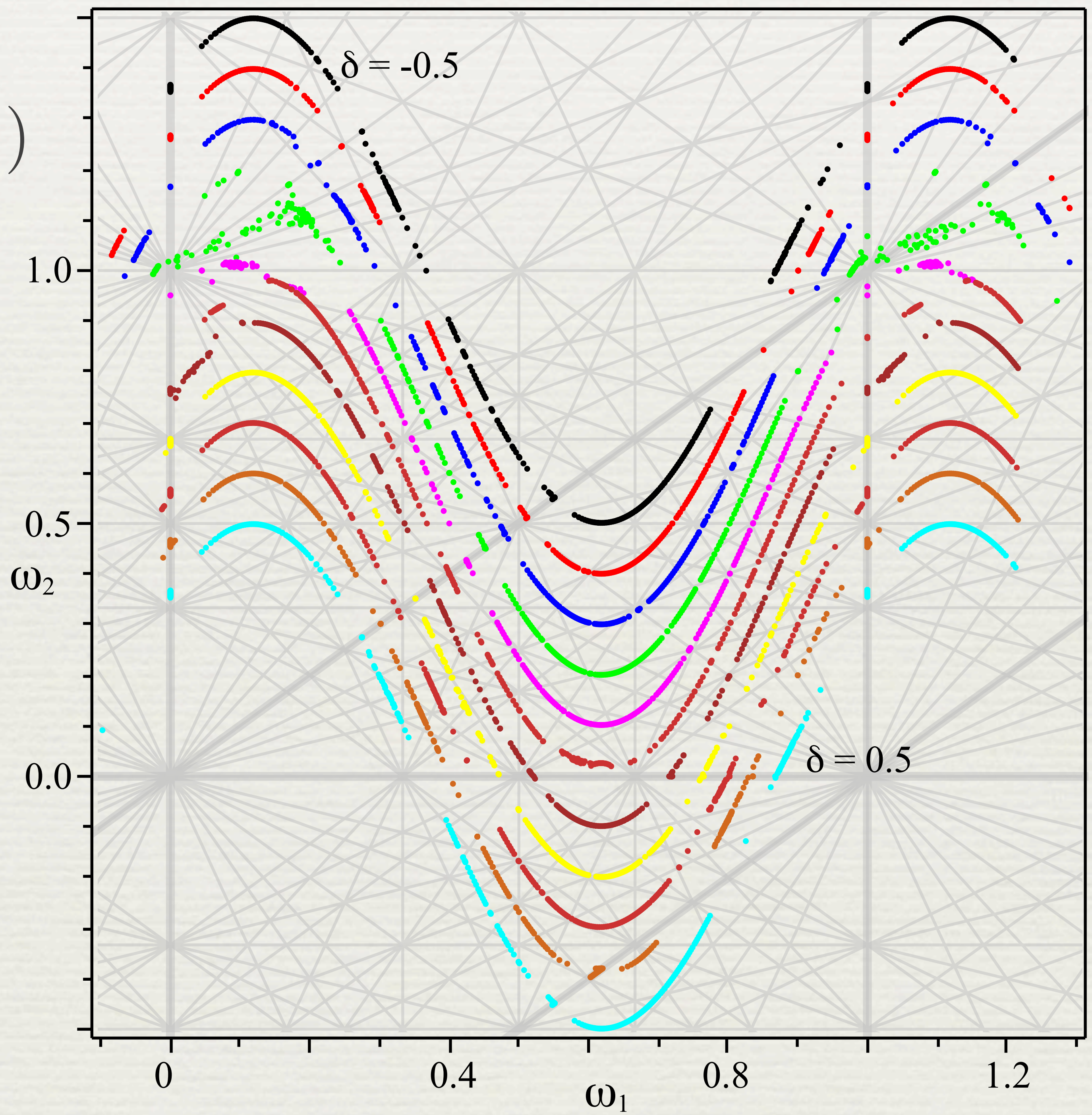
$\therefore$ if there are no tori in $[0,1+\Delta_{max}]$, there are none.

Periodicity in z

Periodic structure in vertical: $\Omega(z+1) = \Omega(z)+m$

$$\Omega(z) = (z+\gamma, -\delta + \lambda \sin^2(\pi z))$$

$$\lambda = 1\,, \quad \gamma = \tfrac{1}{2}(\sqrt{5}-1)$$



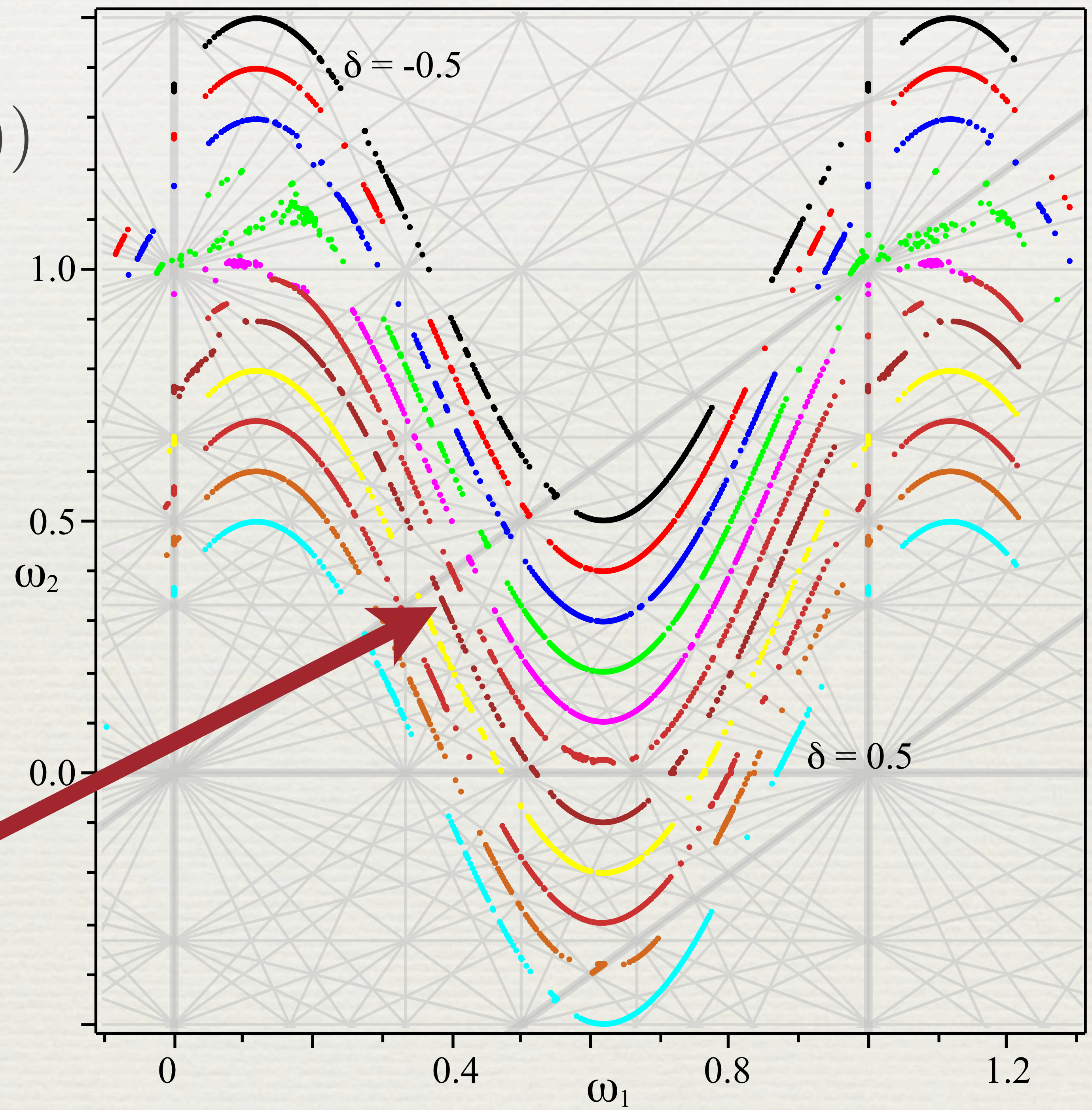
$\varepsilon = 0.01$

Periodicity in z

Periodic structure in vertical: $\Omega(z+1) = \Omega(z)+m$

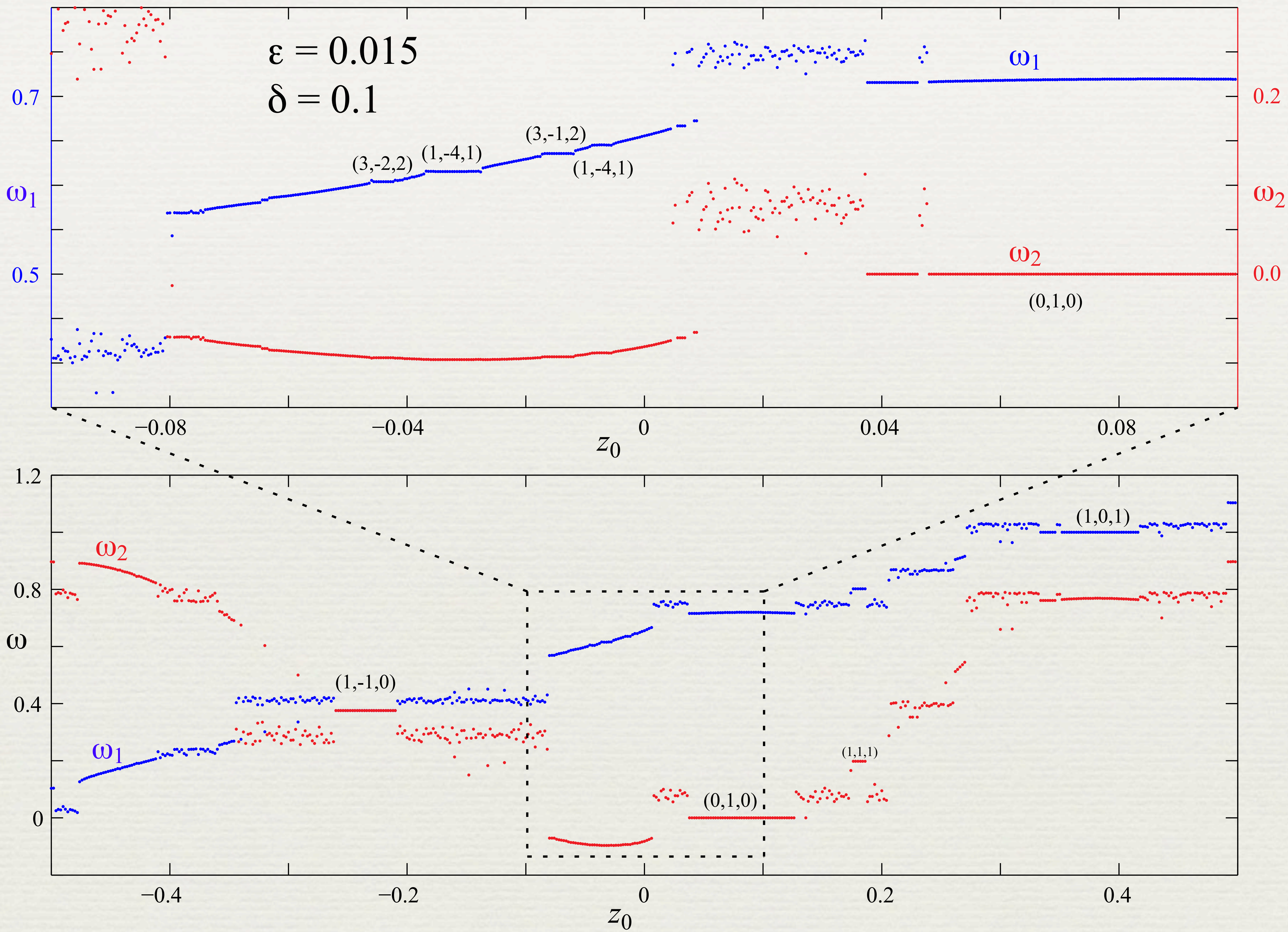
$$\Omega(z) = (z + \gamma, -\delta + \lambda \sin^2(\pi z))$$

$$\lambda = 1\,, \quad \gamma = \tfrac{1}{2}(\sqrt{5} - 1)$$

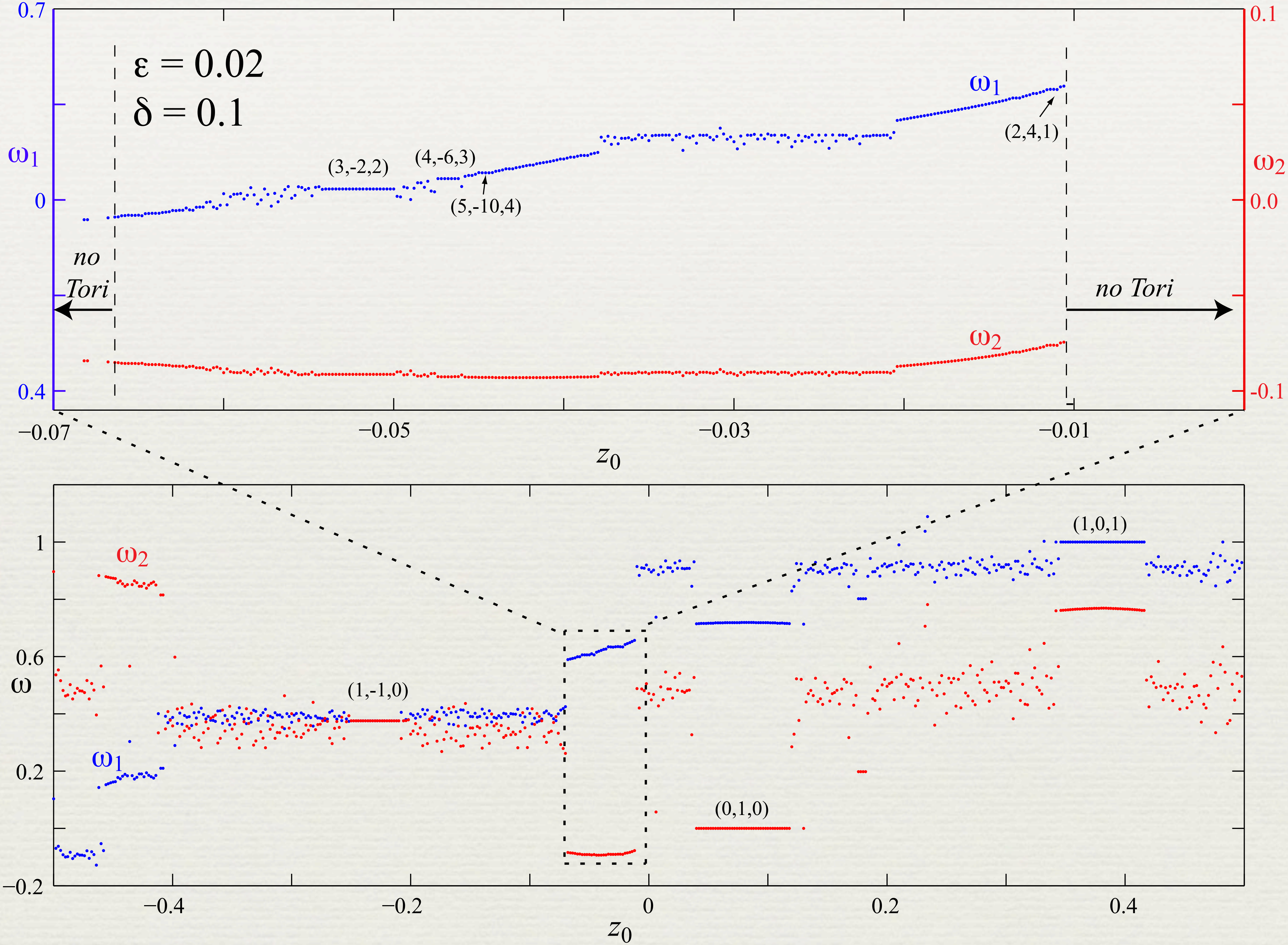


δ =0.1

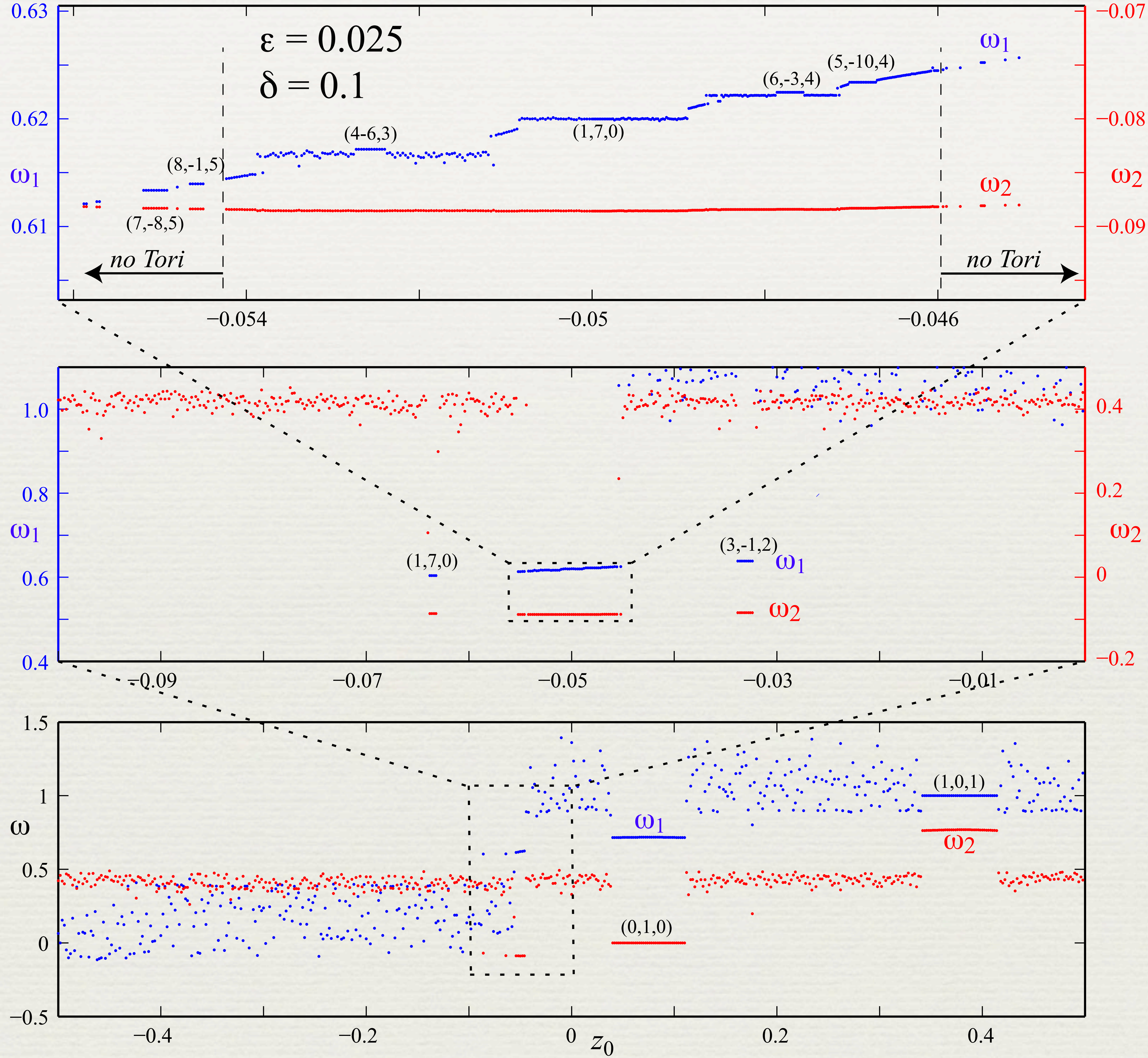
ε = 0.01



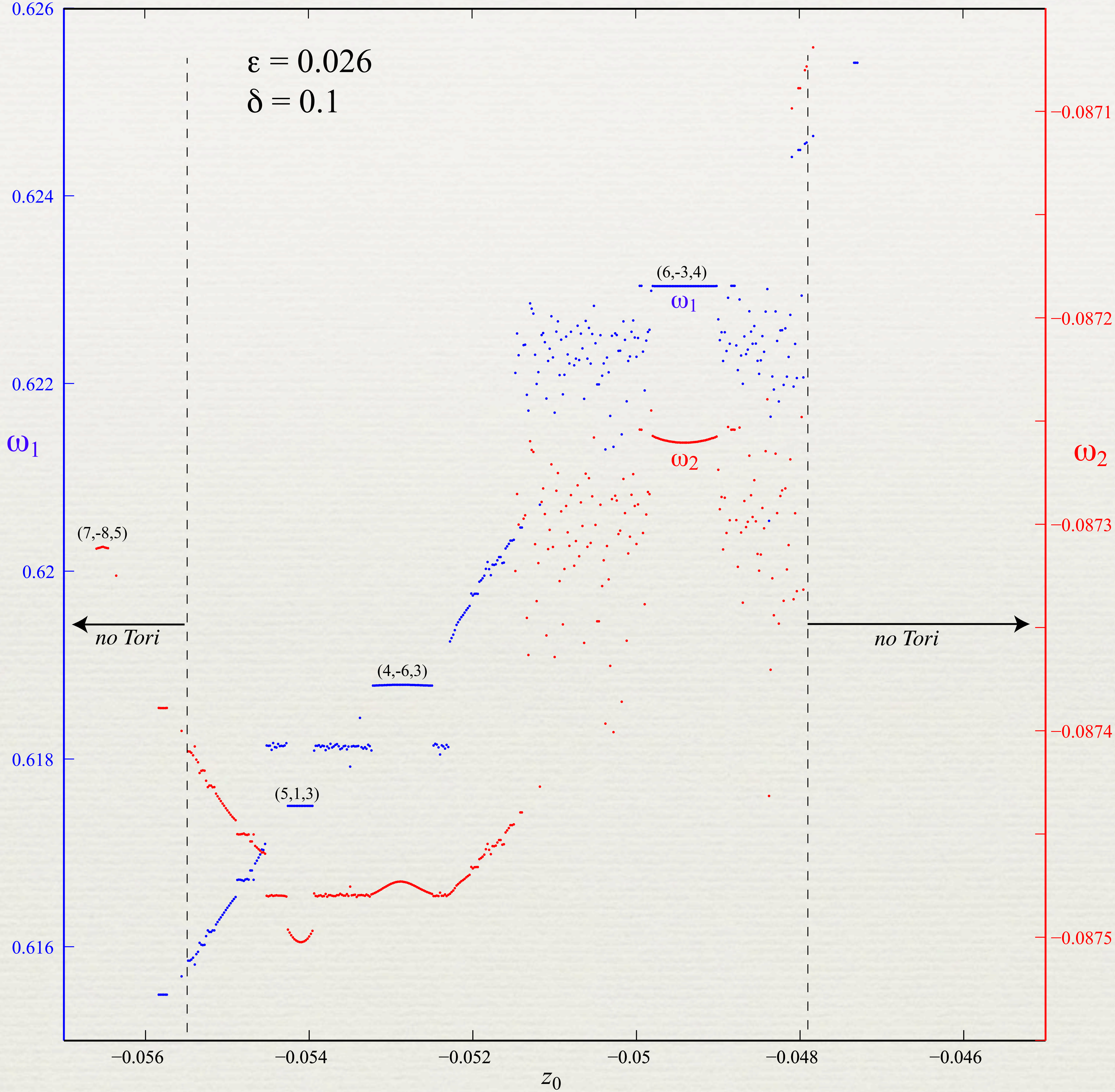
ε = 0.015
δ = 0.1
ω1
ω2
0.7
0.5
0.2
0.0
(3,-2,2)
(1,-4,1)
(3,-1,2)
(1,-4,1)
(0,1,0)
−0.08
−0.04
0
0.04
0.08
z0
1.2
0.8
0.4
0
ω
(1,-1,0)
(0,1,0)
(1,1,1)
(1,0,1)
−0.4
−0.2
0
0.2
0.4
z0

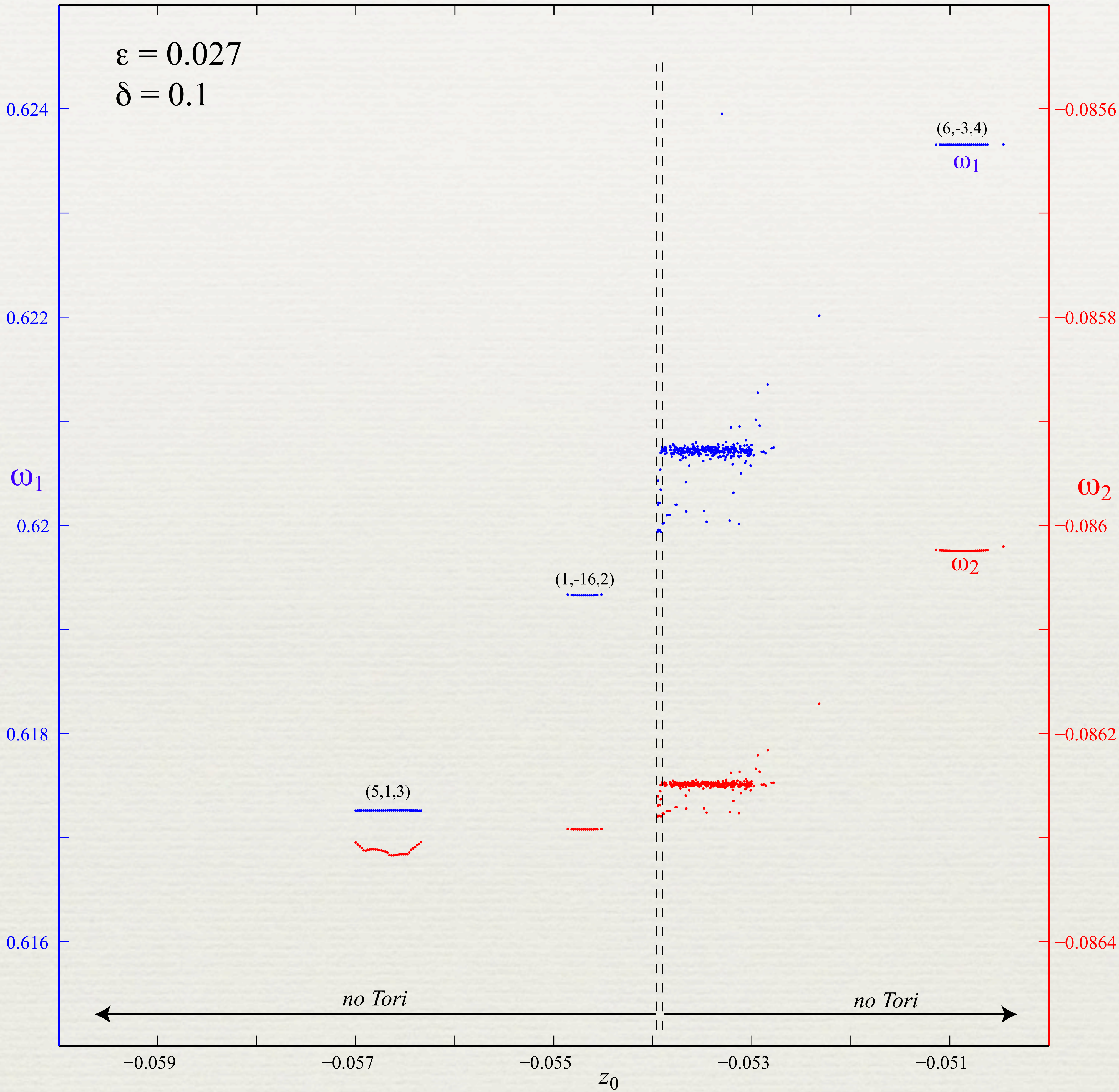


ε = 0.02
δ = 0.1
ω1
ω2
(3,-2,2)
(4,-6,3)
(5,-10,4)
(2,4,1)
no Tori
no Tori
0.7
0
0.4
0.1
0.0
-0.1
−0.07
−0.05
−0.03
−0.01
z0
(1,-1,0)
(0,1,0)
(1,0,1)
ω
1
0.6
0.2
−0.2
−0.4
−0.2
0
0.2
0.4

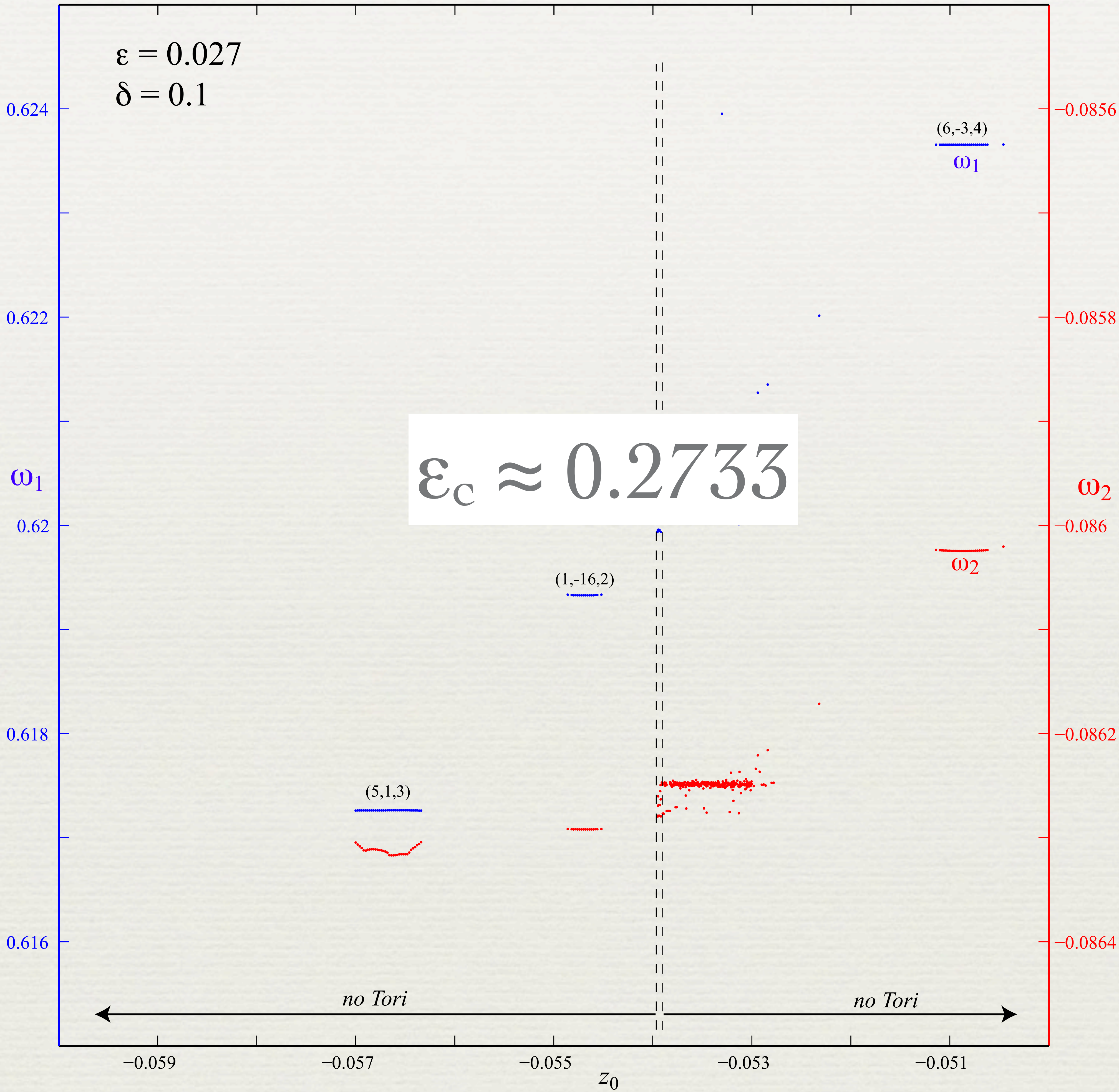


ε = 0.025
δ = 0.1
ω1
ω2
(5,-10,4)
(6,-3,4)
(4-6,3)
(1,7,0)
(8,-1,5)
(7,-8,5)
no Tori
no Tori
0.63
0.62
0.61
−0.07
−0.08
−0.09
−0.054
−0.05
−0.046
1.0
0.8
0.6
0.4
0.2
0
−0.2
−0.09
−0.07
−0.05
−0.03
−0.01
(1,7,0)
(3,-1,2)
1.5
1
0.5
0
−0.5
ω
−0.4
−0.2
0
0.2
0.4
z_0
(1,0,1)
(0,1,0)


ε = 0.026
δ = 0.1
0.626
0.624
0.622
0.62
0.618
0.616
ω1
−0.0871
−0.0872
−0.0873
−0.0874
−0.0875
ω2
(6,-3,4)
ω1
ω2
(7,-8,5)
no Tori
no Tori
(4,-6,3)
(5,1,3)
−0.056
−0.054
−0.052
−0.05
−0.048
−0.046
z0



ε = 0.027
δ = 0.1
(6,-3,4)
ω1
(1,-16,2)
ω2
(5,1,3)
no Tori
no Tori
ω1
ω2
0.624
0.622
0.62
0.618
0.616
−0.0856
−0.0858
−0.086
−0.0862
−0.0864
−0.059
−0.057
−0.055
−0.053
−0.051
z0

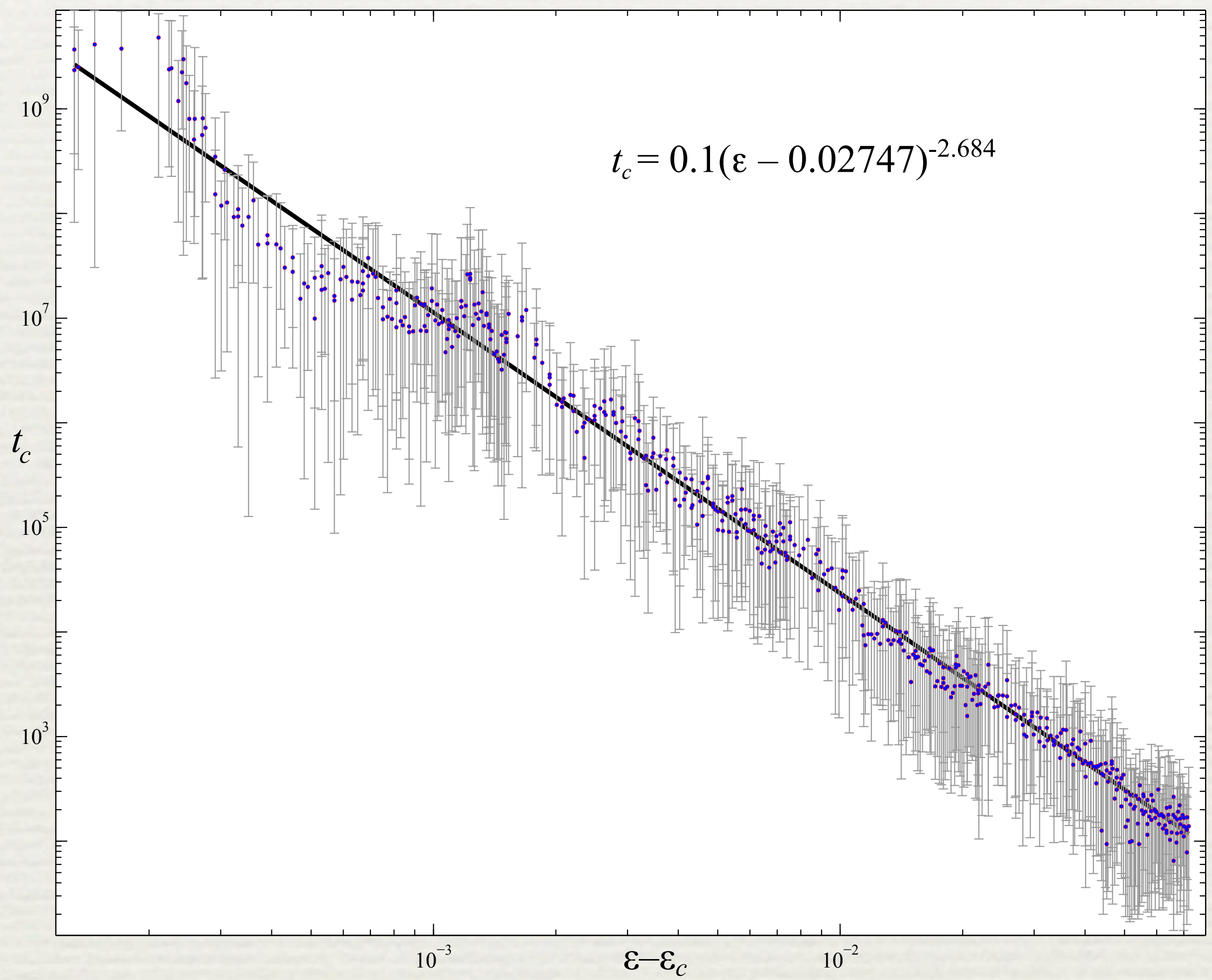


ε = 0.027
δ = 0.1
$\varepsilon_c \approx 0.2733$
(6,-3,4)
ω_1
ω_2
(1,-16,2)
(5,1,3)
no Tori
no Tori
0.624
0.622
0.62
0.618
0.616
−0.0856
−0.0858
−0.086
−0.0862
−0.0864
−0.059
−0.057
−0.055
−0.053
−0.051
z_0

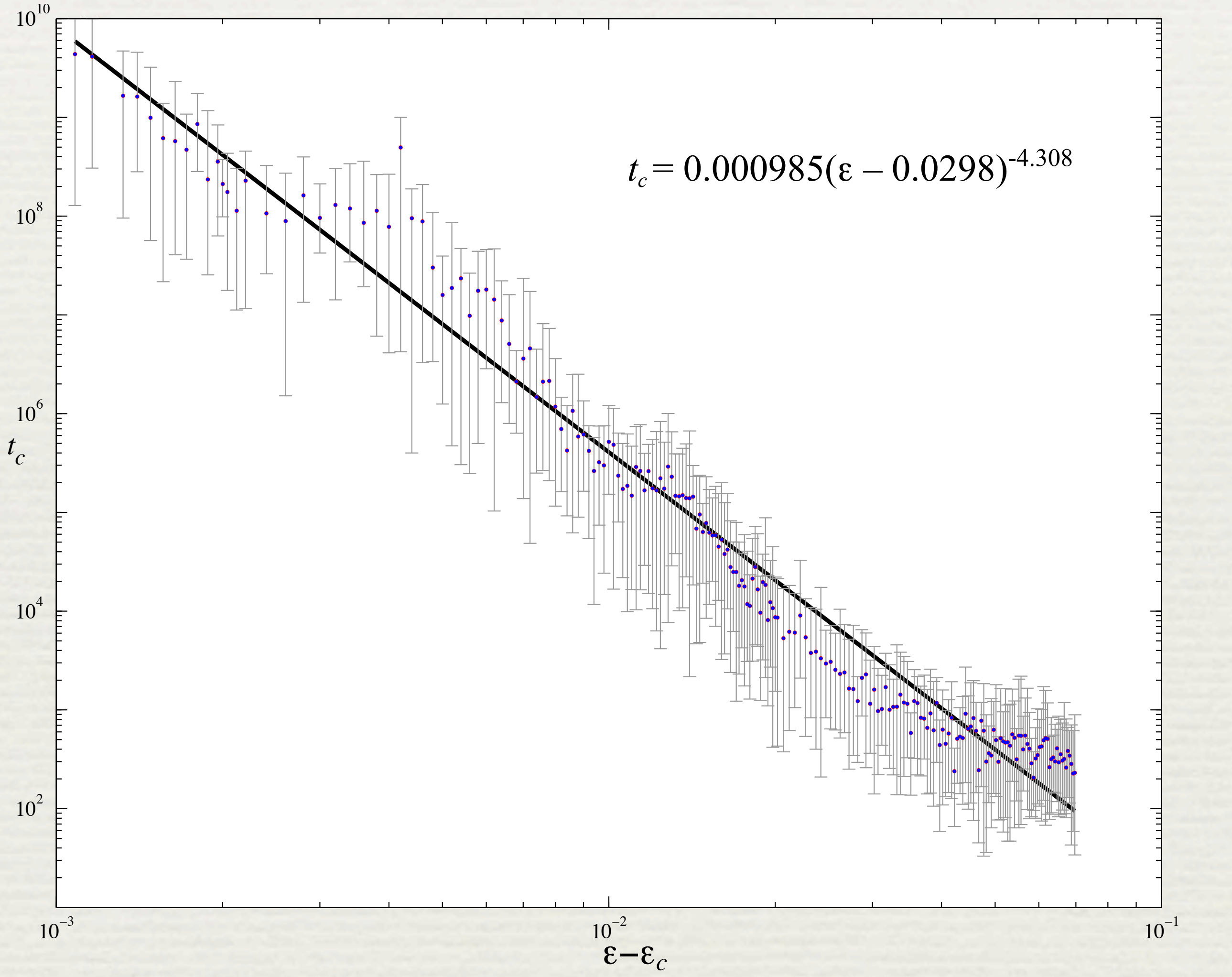
Crossing Time

$$t_c = \min\{t > 0 : |z_t - z_0| \geq 1\}$$

δ = 0.1

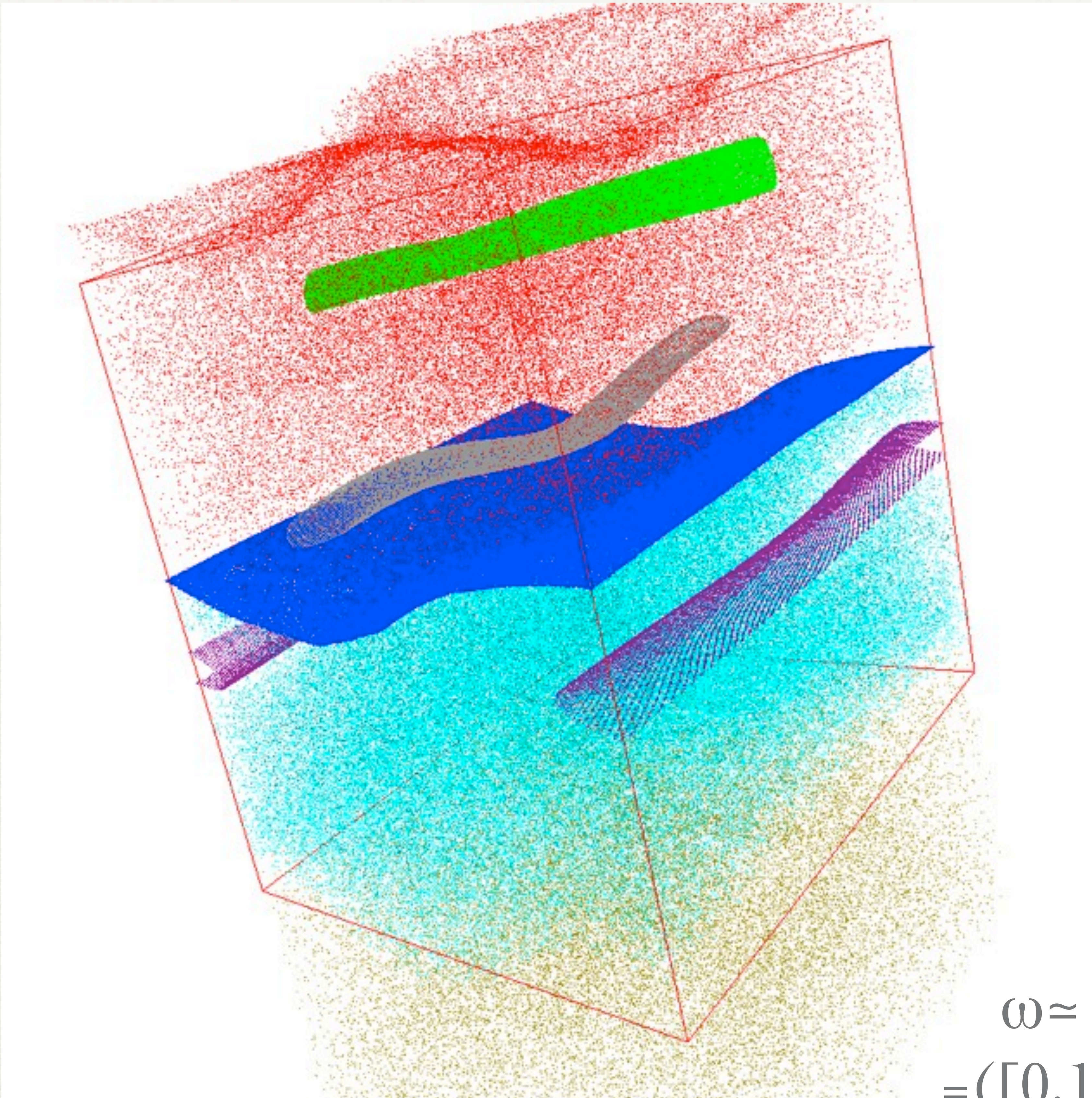


δ = 0.3



10 initial conditions for each ε

The Last Torus?



$\delta = 0.1$

$\varepsilon = 0.02725$

$z_0 = -0.0560$

$\omega \approx (0.618681, -0.085983)$

$= ([0, 1^7, 5, 1^2,], [-1, 1^2, 10, 1^3, 2^2,])$

The Last Torus?

Analog of Greene's Self-similarity?

Analog of the golden mean?

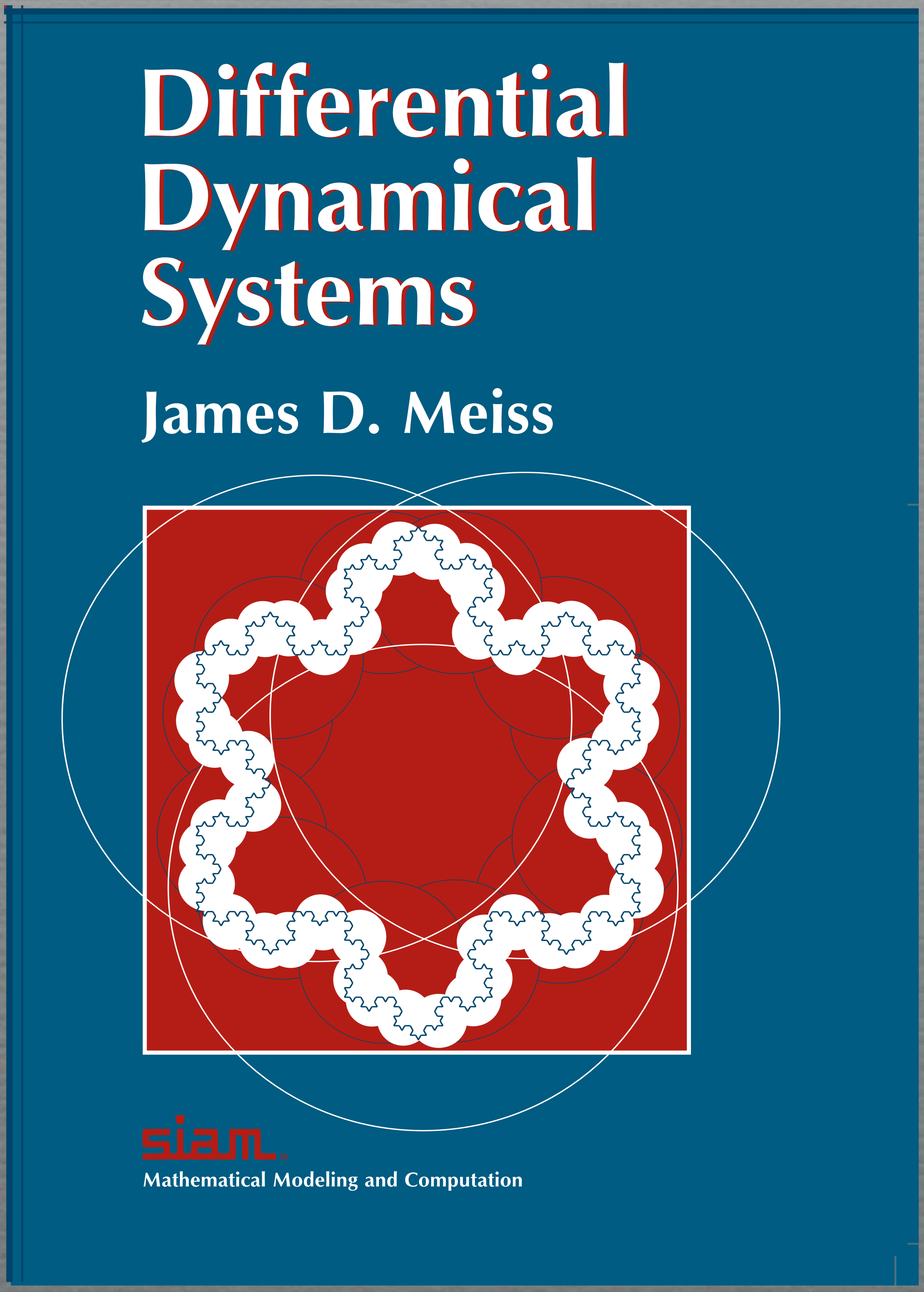
Analog of Birkhoff's 2nd theorem?

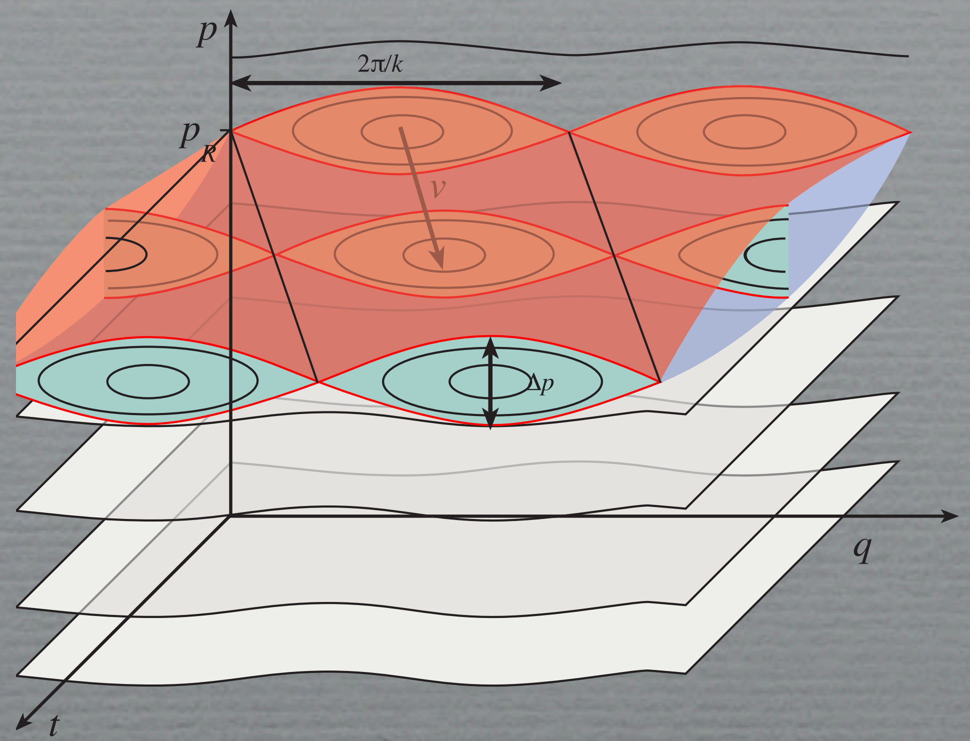
- Rotational tori need not be graphs
- Can one explicitly bound their vertical extent?

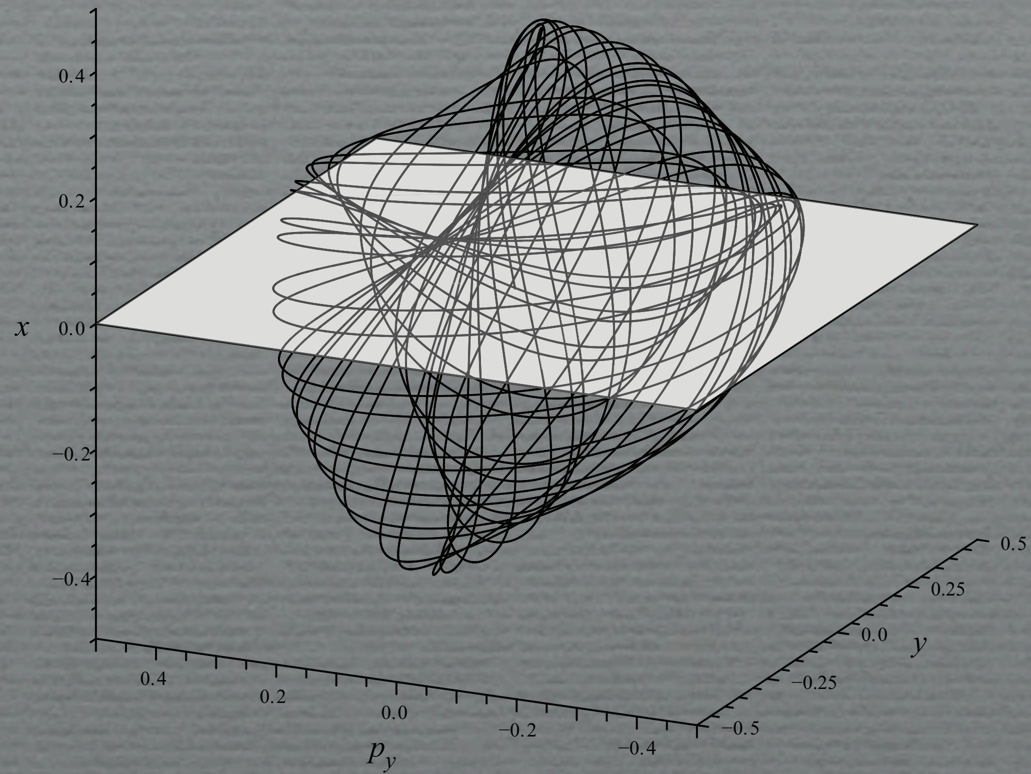
Flux scaling exponent?

Analog of Aubry-Mather: Cantori?

Thanks!







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