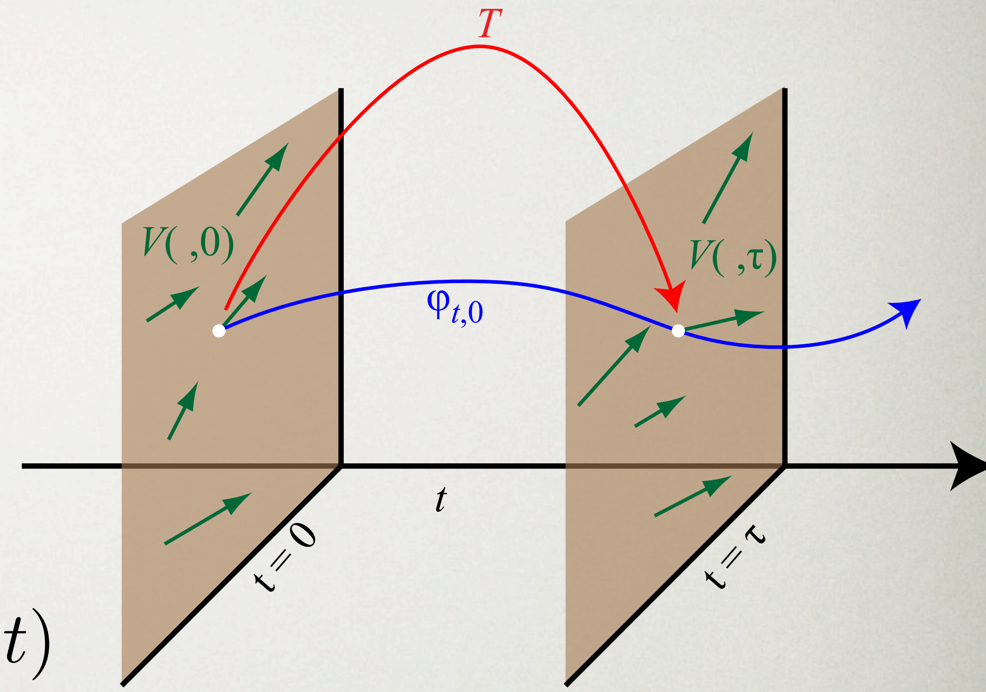
Transport in Time Dependent Flows

an overview

J. D. Meiss
University of Colorado
Boulder

NonAutonomous Dynamics

- General time-dependent vector field $\dfrac{dx}{dt} = V(x,t)$



- Nonautonomous Flow

$$\begin{aligned} \frac{d}{dt}\varphi_{t,t_0}(x) &= V(\varphi_{t,t_0}(x), t) \\ \varphi_{t_0,t_0}(x) &= x \end{aligned}$$

- Time τ Transition Map: $T(x) = \varphi_{t_0+\tau,t_0}(x)$

Example: 2D Fluids

- Assume incompressible:

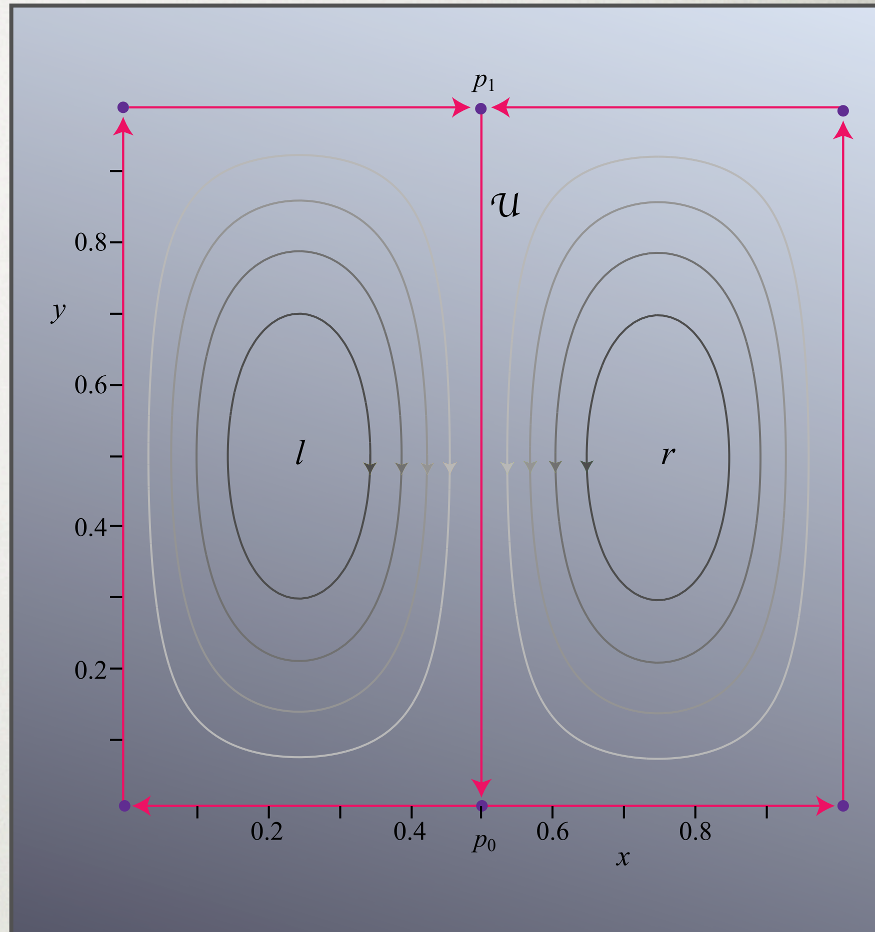
$$\nabla \cdot v = 0$$

- $\Rightarrow$ fluid particle motion is Hamiltonian:

$$v = \hat{z} \times \nabla \psi = \left(-\frac{\partial \psi}{\partial y}, \frac{\partial \psi}{\partial x} \right)$$

- If the velocity is independent of time, ψ is conserved "energy" $\Rightarrow$ motion is along streamlines

$$\begin{aligned} \dot{x} &= -\frac{\partial \psi}{\partial y} \\ \dot{y} &= \frac{\partial \psi}{\partial x} \end{aligned}$$



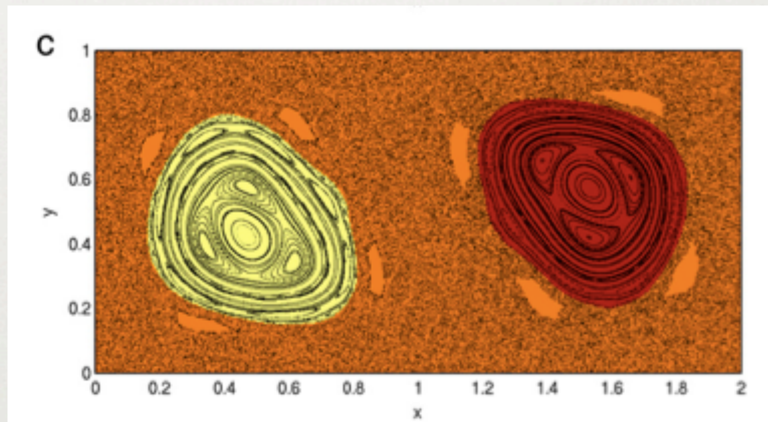
- However if *nonautonomous*, then fluid particles advection may be chaotic.

Oscillating Double Gyre

- Time-Periodic flow with fixed boundaries

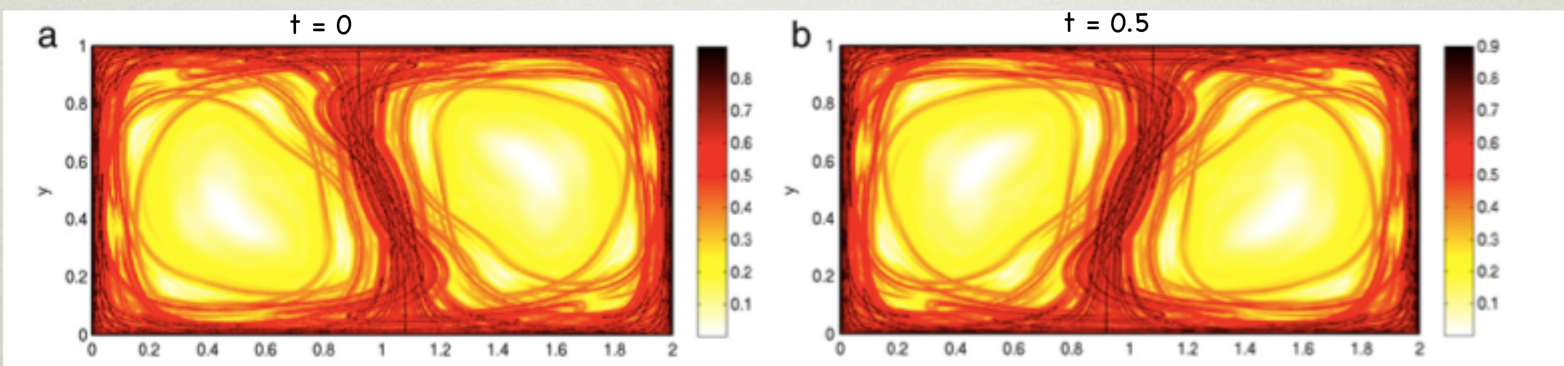
$$\psi(x, y, t) \quad = \quad \pi A \sin[\pi x + \epsilon \pi x(x-2)\sin(\omega t)]\sin(\pi y)$$

Poincaré Section at
$t_n = 2\pi n / \omega$



$A = \varepsilon = 0.25$
$\omega = 2\pi$

Local Expansion Rate (FTLE), $T = 10$



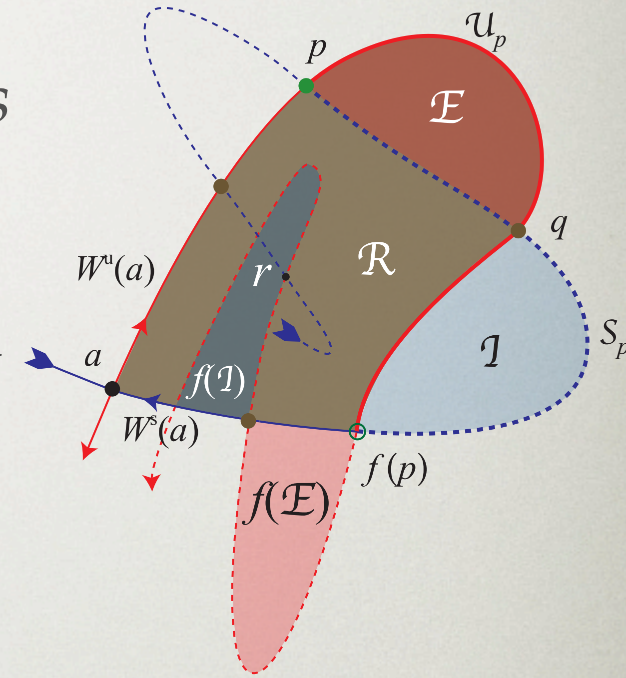
Froyland, G. and K. Padberg (2009). "Almost-invariant sets and invariant manifolds — Connecting probabilistic and geometric descriptions of coherent structures in flows." Physica D **238**: 1507-1523.

Transport

Transport

- The *turnstile mechanism & lobe dynamics*

MacKay, R. S., J. D. Meiss and I. C. Percival (1984). "Transport in Hamiltonian Systems." Physica D **13**: 55-81.
Rom-Kedar, V. and S. Wiggins (1990). "Transport in Two-Dimensional Maps." Arc. Rational Mech. Anal. **109**(3): 239-298.



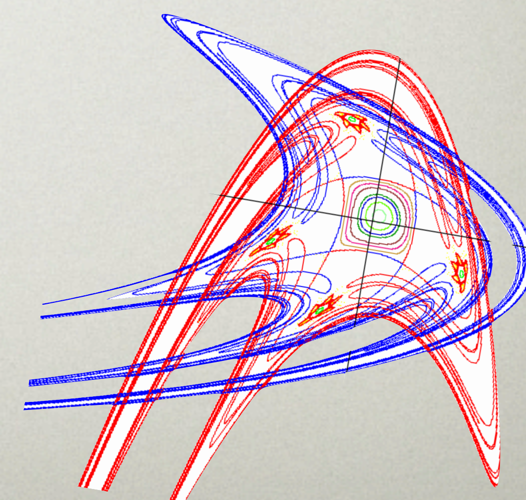
- Nearly autonomous: Melnikov Theory

Sandstede, B., S. Balasuriya, C. K. R. T. Jones and P. Miller (2000). "Melnikov theory for finite-time vector fields." Nonlinearity **13**(4): 1357-1377.

Recall Balasuriya MS 12

- Slow perturbations

Kaper, T. J. and S. Wiggins (1991). "Lobe Area in Adiabatic Hamiltonian Systems." Physica D **51**: 205-212.
Haller, G. and A. C. Poje (1998). "Finite Time Transport in Aperiodic Flows." Physica D **119**: 352-380.



- Transient Perturbations

Malhotra, N. and S. Wiggins (1998). "Geometric structures, lobe dynamics, and Lagrangian transport in flows with aperiodic time-dependence, with applications to Rossby wave flow." J. Non. Sci. **8**: 401–456.
Mosovsky, B. A. and J. D. Meiss (2011). "Transport in Transitory Dynamical Systems." Siam J. Dyn. Sys. **10**(1): 35-65.

Recall Mosovsky MS 12

Recall Mosovsky MS12

Transitory Dynamics

- Past and Future autonomous dynamics:

$$\dot{x} = V(x,t)\ , \quad V(x,t) = \begin{cases} P(x) & t < 0 \\ F(x) & t > \tau \end{cases}$$

- for a transition time τ.

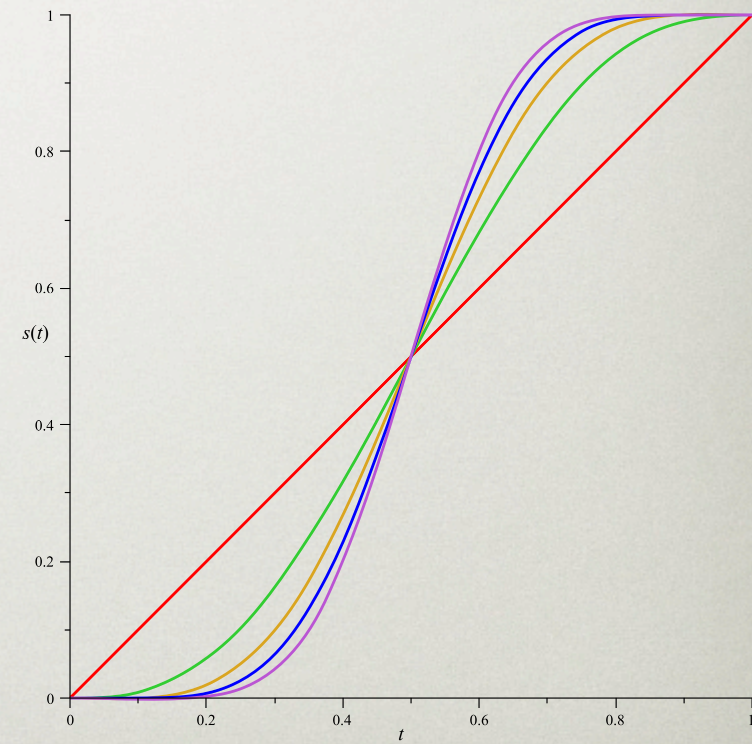
  - For example:

$$V(x,t) = (1 - s(t))P(x) + s(t)F(x)$$

  - Transition function

$$s(t) = \begin{cases} 0 & t < 0 \\ 1 & t > \tau \end{cases}$$

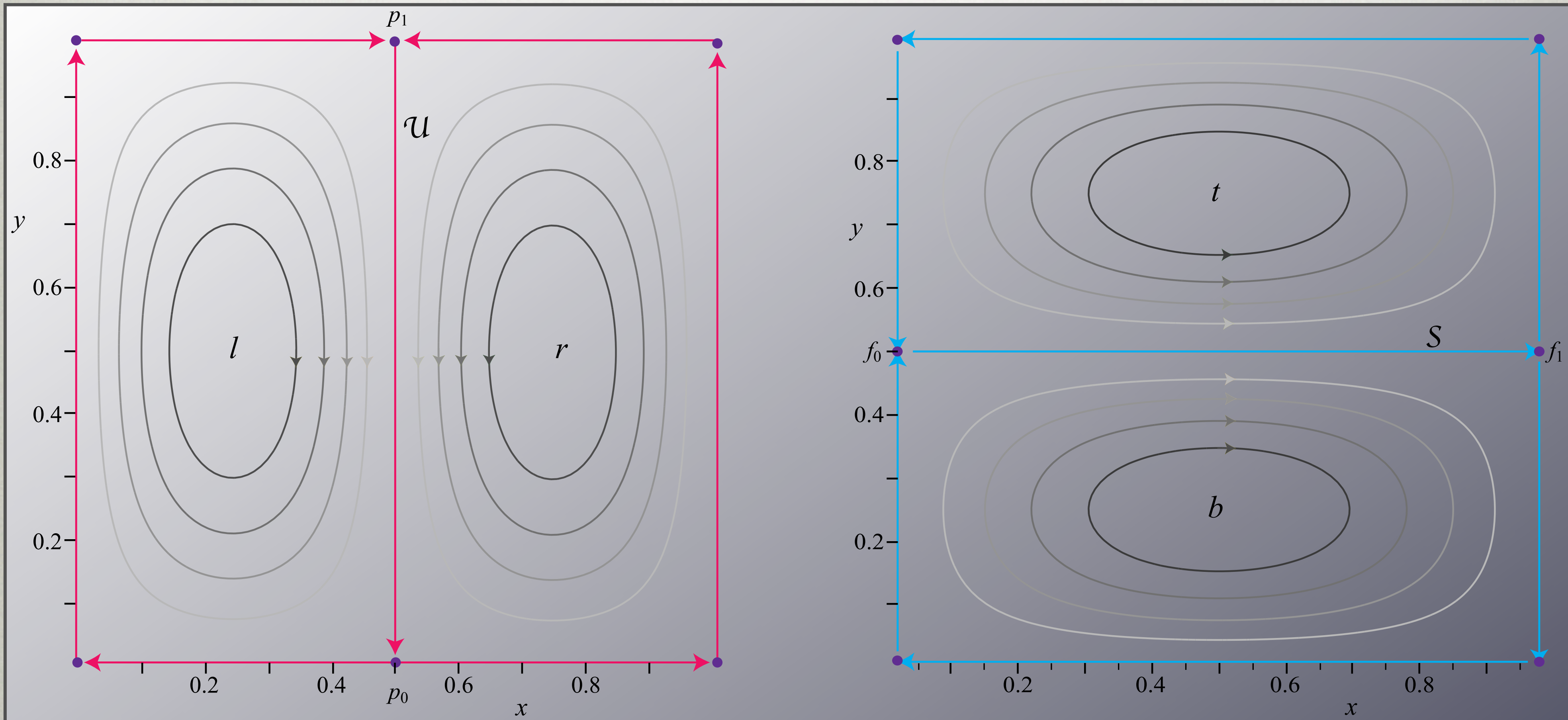
$$s(t) \quad = t^2(3 - 2t)$$



Polynomial s(t)

Rotating Double Gyre

$$\psi_P(x,y) = \sin(2\pi x)\sin(\pi y) \qquad \psi_F(x,y) = \sin(\pi x)\sin(2\pi y)$$

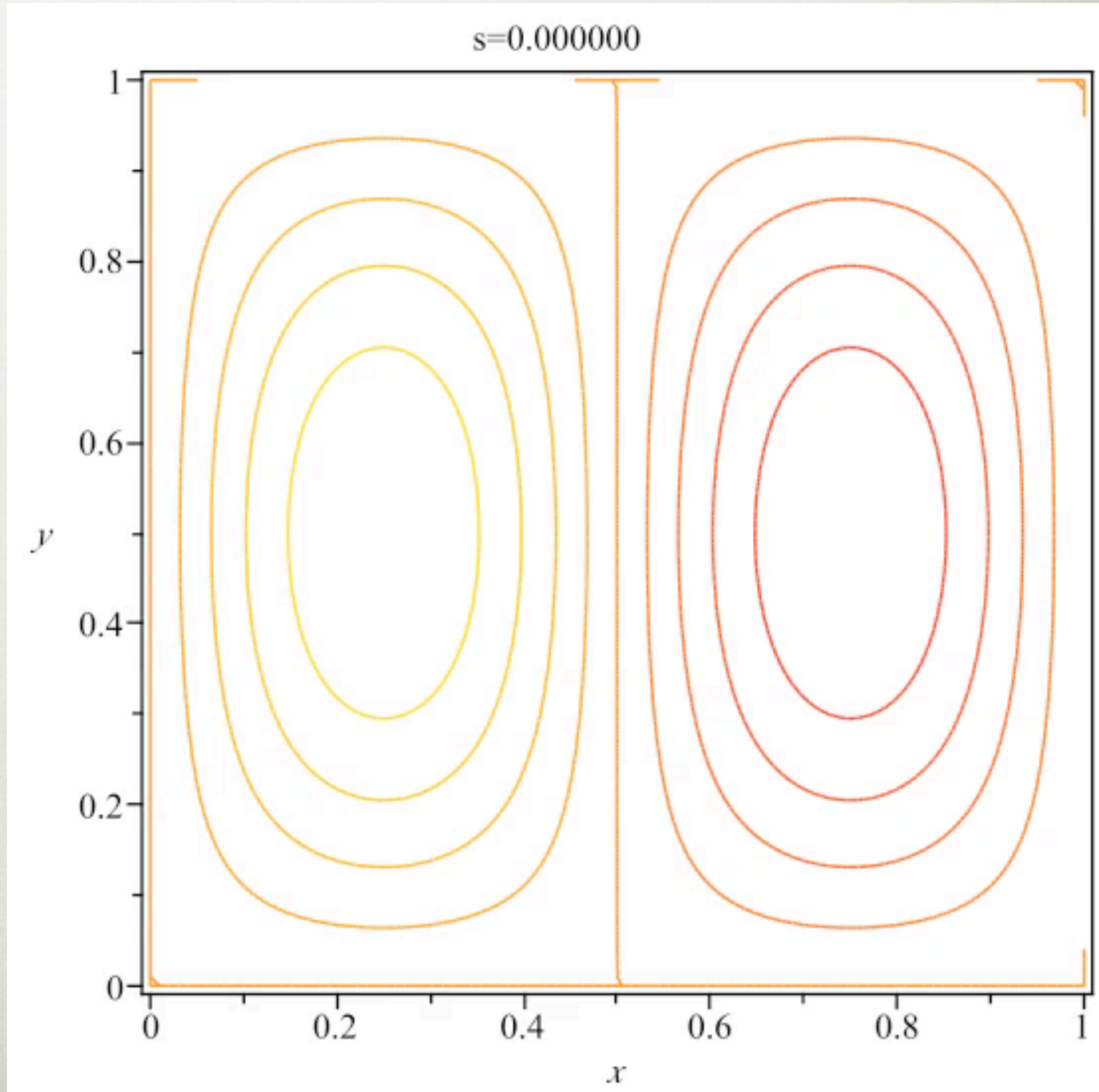


$$\dot{x} = -\frac{\partial}{\partial y}\psi\ , \quad \dot{y} = \frac{\partial}{\partial x}\psi \qquad \psi = (1 - s(t))\psi_P + s(t)\psi_F$$

Rotating Double Gyre

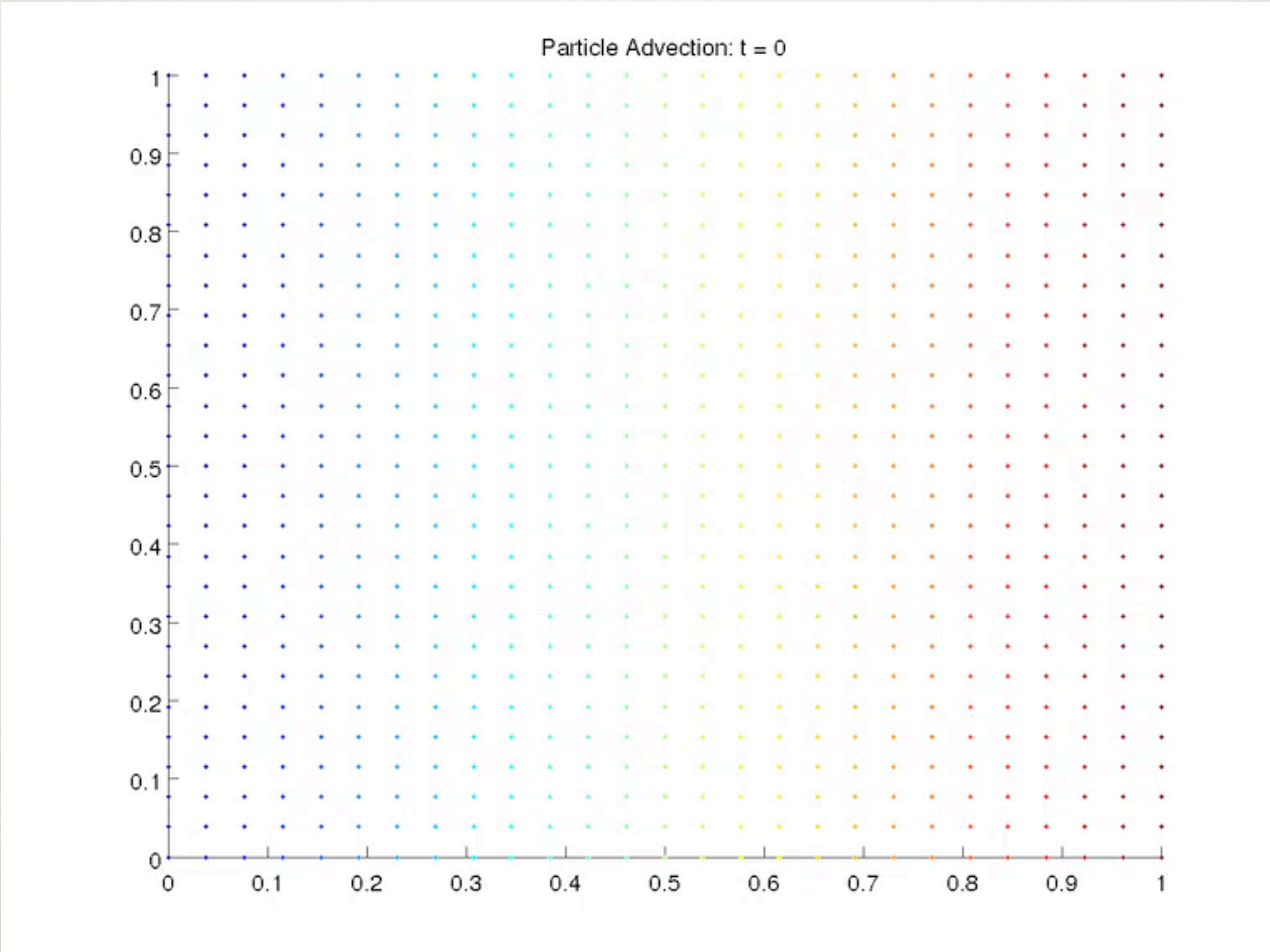
$$\psi_P(x,y) = \sin(2\pi x)\sin(\pi y)$$
$$\psi_F(x,y) = \sin(\pi x)\sin(2\pi y)$$

$$\psi(x,y,s) = \quad (1-s)\psi_P(x,y) + s\psi_F(x,y)$$



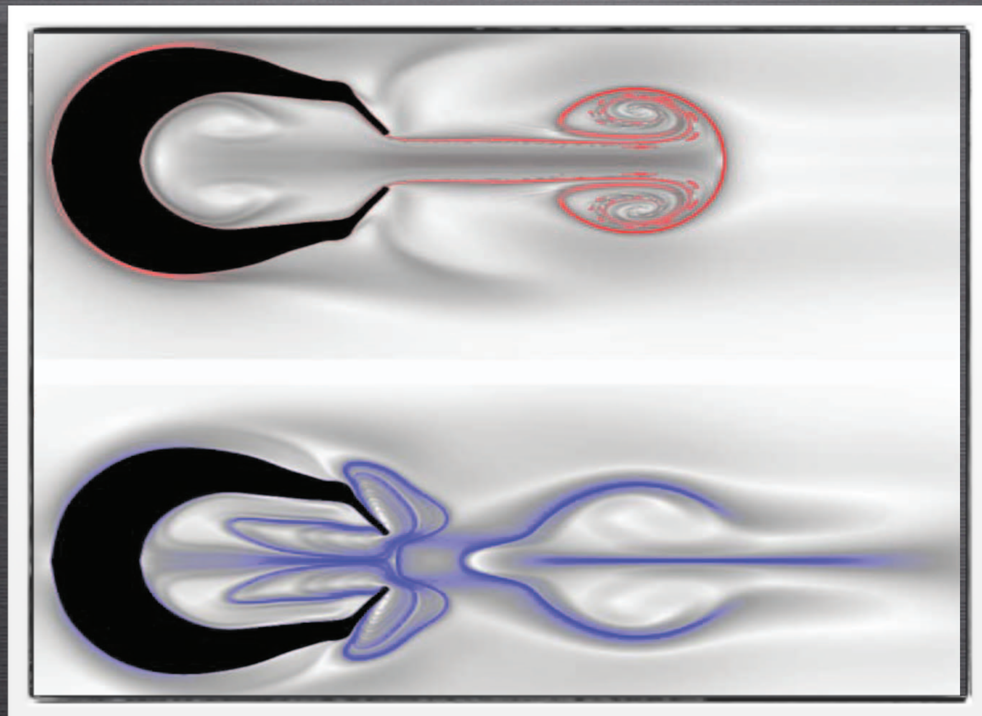
Rotating Double Gyre

$\tau = 1.0$



Particle trajectories do not lie on streamlines!

CFD code for Sarsia Tubulosa Jellyfish



Lagrangian Coherent Structures

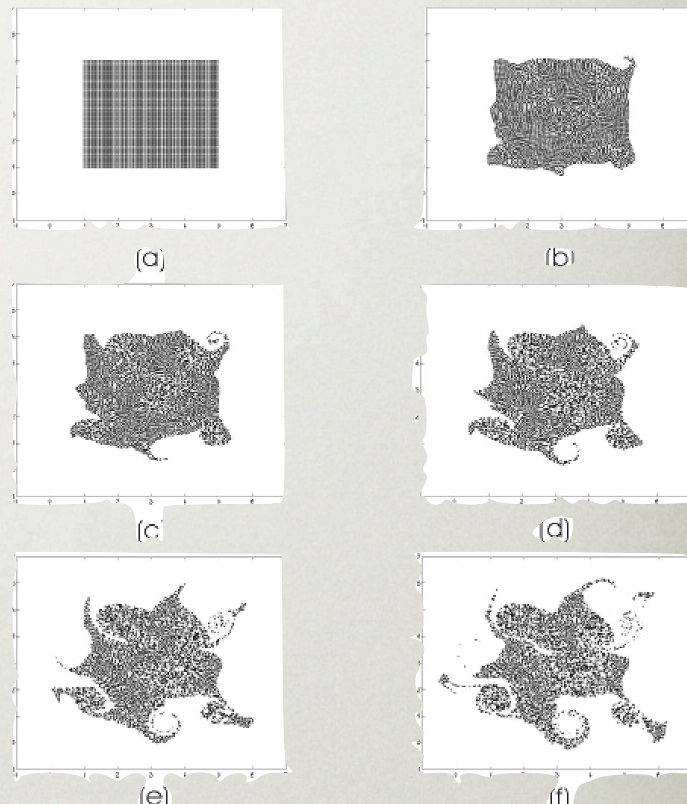
Lipinski, D. and K. Mosheni (2010). “A ridge tracking algorithm and error estimate for efficient computation of Lagrangian coherent structures.” Chaos **20**: 017504.

Lagrangian Coherent Structures

- Coherent trajectory patterns on a finite time interval
  - boundary is codimension-one & "simple"
  - ≈ invariant under nonautonomous flow
  - may live for finite time
- Hyperbolic boundaries: material lines with locally the longest or shortest stability or instability time (Haller & Yuan 2000)
- Almost Invariant Sets

$$\mu(A \cap \varphi_{t+\tau,t}(A)) \approx \mu(A)$$

  - In the sense of measure

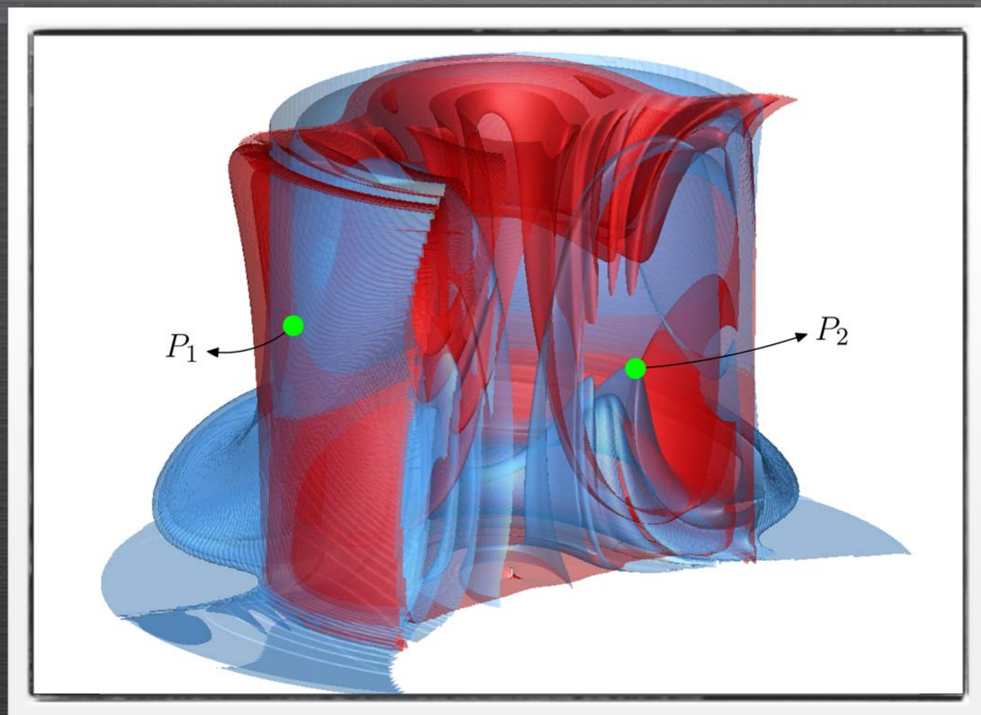


quasi-geostrophic flow
(Haller & Yuan 2000)

Finding LCS

- Finite Time/Size Lyapunov Exponents:
  - Pierrehumbert, R. T. (1991). “Large-Scale Horizontal Mixing in Planetary Atmospheres.” Phys. Fluids 3A(5): 1250-1260.
  - Liu, M., F. J. Muzzio and R. L. Peskin (1994). “Quantification of Mixing in Aperiodic Chaotic Flows.” Chaos, Solitons and Fractals 4(6): 869-893.
- Distinguished Hyperbolicity
  - Haller, G. (2001). “Distinguished material surfaces and coherent structures in three-dimensional fluid flows.” Physica D 149(4): 248-277.
  - Jiménez Madrid, J. A. and A. M. Mancho (2009). “Distinguished Trajectories in Time Dependent Vector Fields.” Chaos 19: 013111.
- Almost Invariant Sets
  - Froyland, G. (2005). “Statistically optimal almost-invariant sets.” Physica D 200(3-4): 205-219.
  - Froyland, G. and K. Padberg (2009). “Almost-invariant sets and invariant manifolds” Physica D 238: 1507-1523.

Convection model with random time dependence



FTLE

Lekien, F., S. C. Shadden and J. E. Marsden (2007). "Lagrangian coherent structures in n-dimensional systems." J. Math. Phys. **48**(6): 065404.

FTLE or FSLE

- Finite Time (Nese 1989):

Nese, J. M. (1989). "Quantifying local predictability in phase space." Physica D: Nonlinear Phenomena **35**(1-2): 237-250.
Haller, G. (2001). "Distinguished material surfaces and coherent structures in three-dimensional fluid flows." Physica D **149**(4): 248-277.

- Finite Size (Aurell et al 1997):

Aurell, E., G. Boffetta, et al. (1997). " Predictability in the large: An extension of the concept of Lyapunov exponent." J. Phys. A: Math. Gen. **30**: 1–26.
Keane, R. J., P. L. Read and G. P. King (2010). "Effectiveness of stirring measures in an axisymmetric rotating annulus flow." Physica D: **239**(10): 675-683.

- FTLE Ridges (Shadden et al 2005):
  - A ridge of the FTLE field may be nearly invariant and have low flux

Shadden, S. C., F. Lekien and J. E. Marsden (2005). "Definition and properties of Lagrangian coherent structures from finite-time Lyapunov exponents in two-dimensional aperiodic flows." Phys. D **212**(3-4): 271-304.

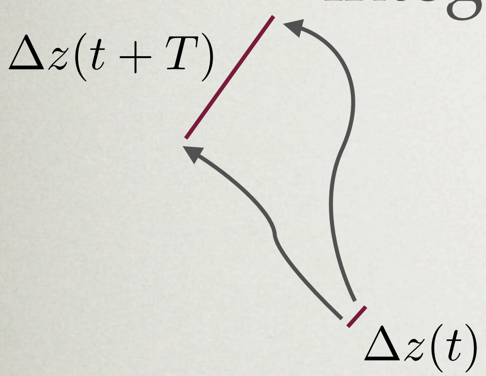
- Finite Time Manifolds for nearly autonomous case (Sandstede et al 2000) $V(x,t) = V_0(x) + \varepsilon V_1(x,t)$
  - Integration time $O(\ln \varepsilon)$ to compute splitting by Melnikov theory.

Sandstede, B., S. Balasuriya, C. K. R. T. Jones and P. Miller (2000). "Melnikov theory for finite-time vector fields." Nonlinearity **13**(4): 1357-1377.

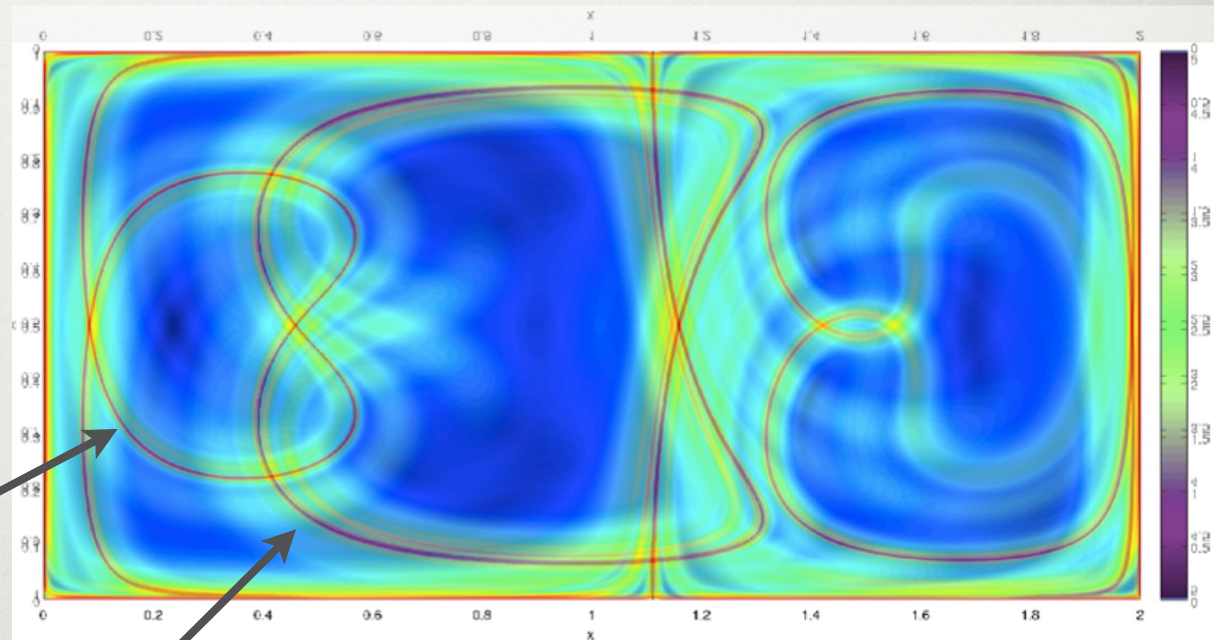
Finite Time Lyapunov Exponents

FTLE at time t,
integrate forward to time t+T

"FTLE"



$$\lambda(z,t) = \frac{1}{T}\ln\left(\frac{|\Delta z(t+T)|}{|\Delta z(t)|}\right)$$



Ridges ≈ Stable Manifolds

Oscillating Double Gyre   ε = 0.25, T = 1.3
Grid of 1001x500 points

Similarly: backward time integration shows unstable manifolds

Finite Time Lyapunov Exponents

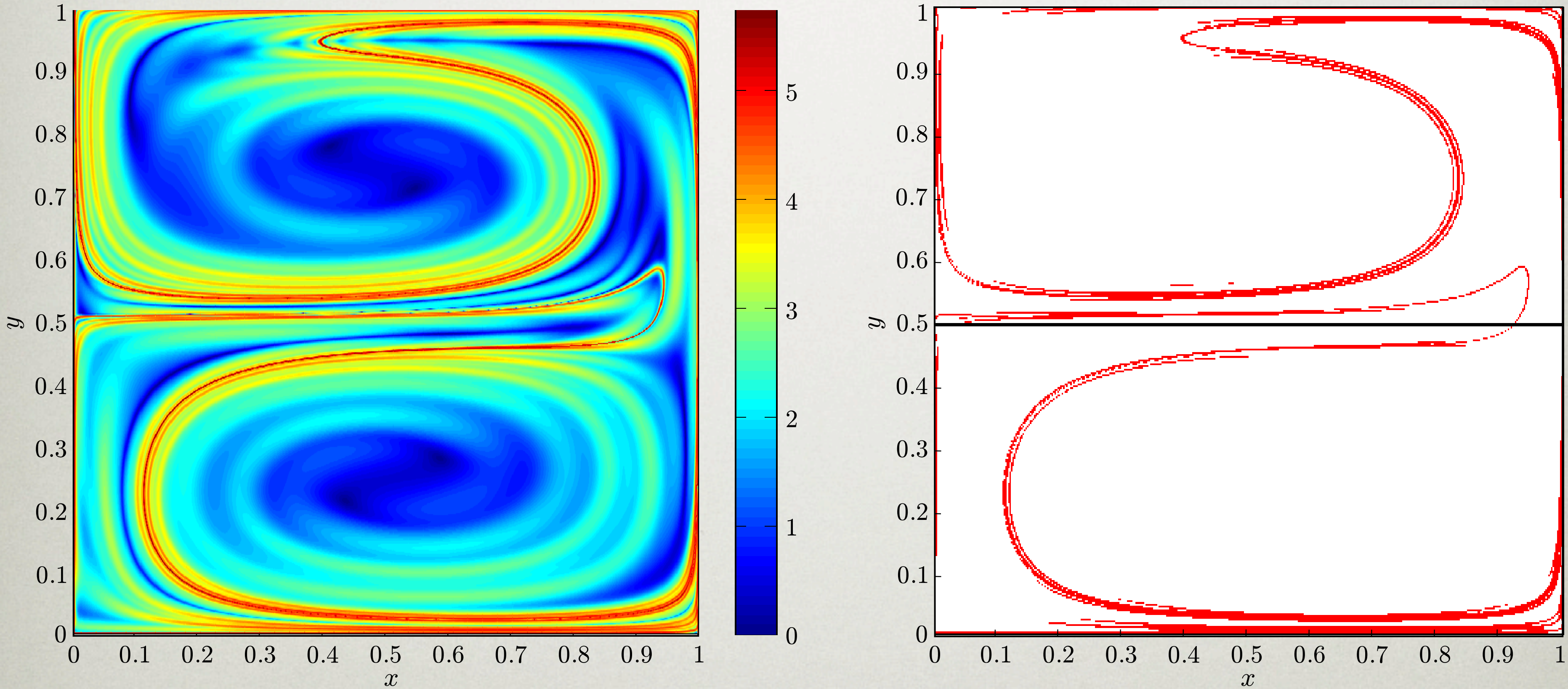
- Singular values of the linearized flow (eigenvalues of the Cauchy-Green matrix) $C(x,s,t) = D\varphi_{s,t}(x)^T D\varphi_{s,t}(x)$

$$\lambda(x,t) = \frac{1}{2T} \log\left[\mathrm{Eig}_{max} C(x, t+T, t)\right]$$

- For T→ ∞: Invariant surfaces normal to ∇λ
- Ridges: curves of flow of ∇λ transverse to direction of minimum curvature
- Low flux if *T "large enough"*

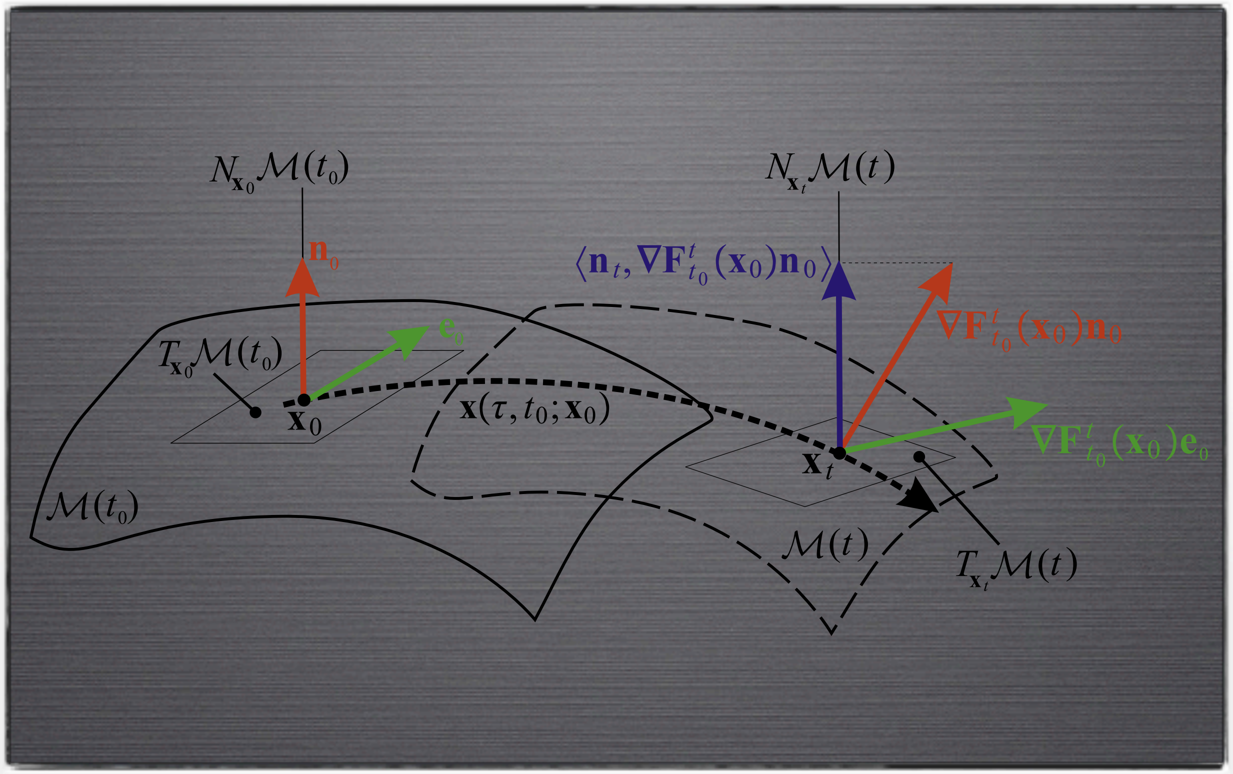
Limitations of FTLE

- Shear causes ridges
- Ridges break-up
- Strength ≠ Low Flux?
- Excess computation*
- Time-scale dependent



Backwards FTLE at $t = \tau = 0.8$ $T = -1.2$

* Ridge tracking may help: see e.g. Lipinski, D. and K. Mosheni (2010). "A ridge tracking algorithm and error estimate for efficient computation of Lagrangian coherent structures." Chaos **20**: 017504.

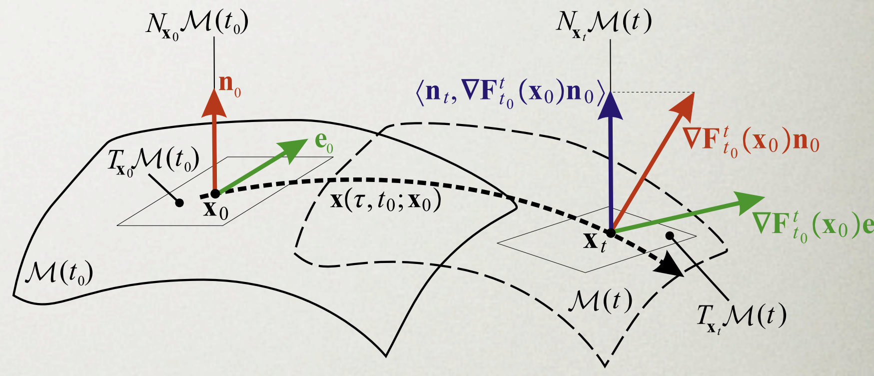


Local Hyperbolicity

Haller, G. and T. Sapsis (2010). "Localized Instability and Attraction along Invariant Manifolds." SIAM J. App. Math. **9**(2): 611-633.

Local Hyperbolicity

- The boundary of a hyperbolic LCS over a finite time interval *I* is a locally strongest repelling or attracting material surface over I.



- codimension-one surface S: $\mathbb{R}^{n-1} \rightarrow M$
- Transition map $T = \varphi_{t+\tau,t}$
- projected normal vector flow:

$$d_\tau = \langle \hat{n}_{t+\tau}, T_* \hat{n}_t \rangle$$

- Normally repelling:

$$d_\tau > e^{a\tau}$$

$$\|T_* \hat{t}\| < d_\tau e^{-b\tau}$$

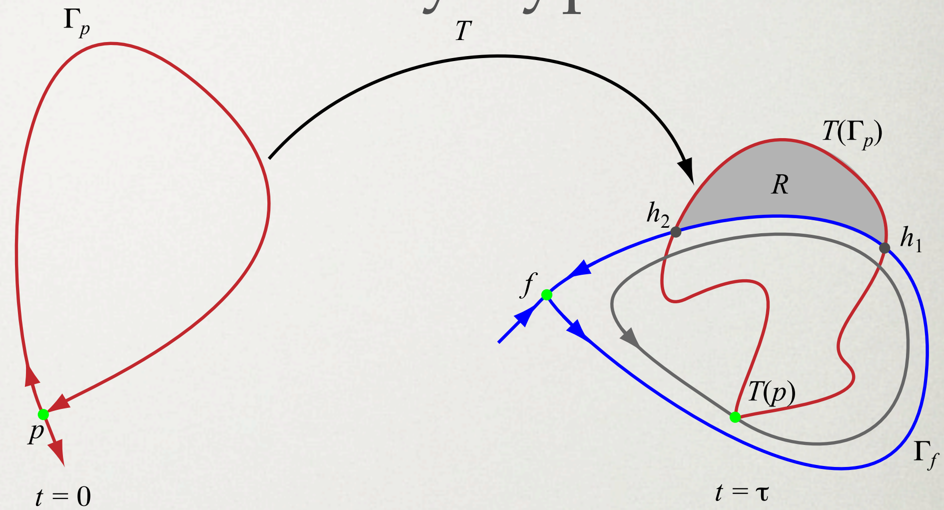
Haller, G. (2000). “Finding finite-time invariant manifolds in two-dimensional velocity fields ” Chaos **10**: 99-108
Haller, G. and T. Sapsis (2010). “Localized Instability and Attraction along Invariant Manifolds.” SIAM J. App. Math. **9**(2): 611-633.
Haller, G. (2011). “A variational theory of hyperbolic Lagrangian Coherent Structures.” Phys. D **240**: 573-598.

Partial Hyperbolicity: Transient Case

- No reason LCS boundaries need be fully hyperbolic
  - $\Lambda \in M \times \mathbb{R}$ invariant if

$$\Lambda_t = \varphi_{t,t_0}(\Lambda_{t_0})$$

  - Transient extension: extend vector field outside finite interval by autonomous $V(x)$



  - For nearly autonomous case: "essentially unique" manifold extensions to o(ε) for V defined only on finite time intervals of order ln(ε)/λ.

Sandstede, B., S. Balasuriya, C. K. R. T. Jones and P. Miller (2000). "Melnikov theory for finite-time vector fields." Nonlinearity **13**(4): 1357-1377.

$$W^u_\tau(\Lambda) = \{x \in M : \lim_{t \to -\infty} |\varphi_{t,\tau}(x) - \Lambda_t| = 0\},$$

  - Unstable manifolds naturally determined by $t \to -\infty$
    - *Backward Hyperbolic* if hyperbolic under past (asymptotically autonomous) flow
  - Stable Manifolds naturally determined by $t \to \infty$
    - *Forward Hyperbolic* if hyperbolic under future (asymptotically autonomous) flow

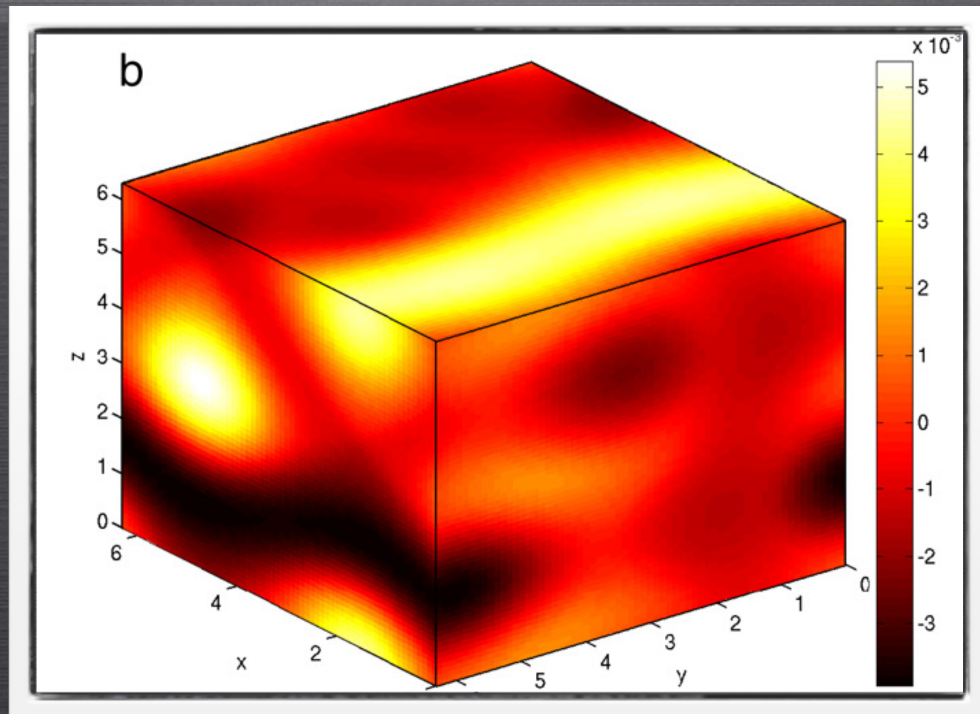
Questions

- Are "normally hyperbolic" surfaces always appropriate LCS boundaries?
  - Are there nonautonomous ≈ versions of invariant circles/tori/cantori?
- Is there an effective way to compute transport?
  - *Action Flux formulas* for lobe volumes (from heteroclinic orbits) for transient case

Mosovsky, B. A. and J. D. Meiss (2011). "Transport in Transitory Dynamical Systems." Siam J. Dyn. Sys. **10**(1): 35-65.

- Can normal hyperbolicity be computed for vector fields given by data instead of models?

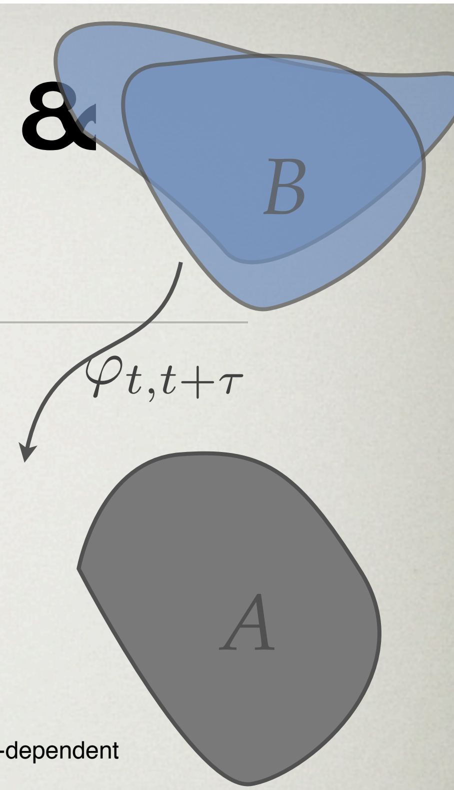
Second Perron-Frobenius Eigenfunction for the ABC flow



Almost Invariant Sets

Froyland, G. and K. Padberg (2009). "Almost-invariant sets and invariant manifolds — Connecting probabilistic and geometric descriptions of coherent structures in flows." Physica D **238**: 1507-1523.

Almost Invariance & Coherence



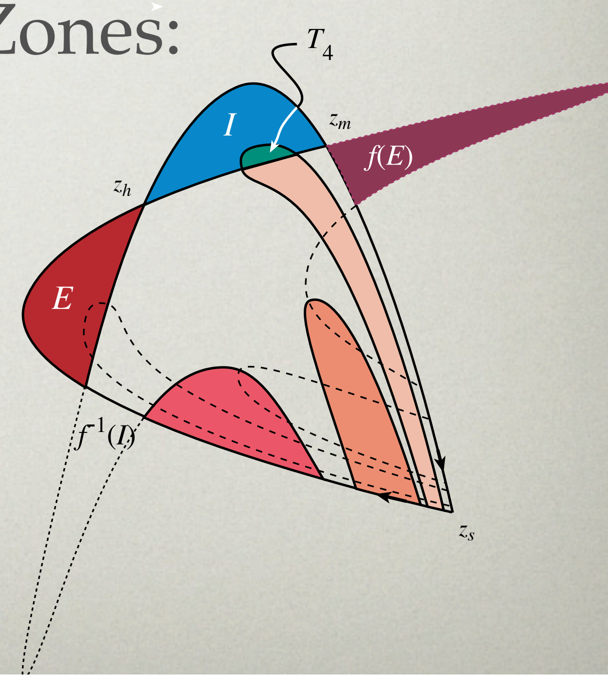
- Sets A & B are *coherent* if

$$\rho_\tau(A,B) = \frac{\mu(A \cap \varphi_{t,t+\tau}(B))}{\mu(A)} > \rho_0$$

  - If A is coherent with itself, it is almost invariant.
  - Minimize escaping flux

Froyland, G., N. Santitissadeekorn and A. Monahan (2010). "Transport in time-dependent dynamical systems: Finite-time coherent sets." Chaos **20**(4): 043116.

- Minimizing escaping flux: Resonance Zones:
  - Enclosed by segments of nearly coincident stable & unstable manifolds
  - Flux is independent of switching point from W^u to W^s
  - For twist maps, minimal flux surfaces are cantori



MacKay, R. S., J. D. Meiss and I. C. Percival (1984). "Transport in Hamiltonian Systems." Physica D **13**: 55-81.

Almost Invariance

- Eigenfunctions of the Perron-Frobenius Operator

$$\mathcal{P}[f] = |\det(D\varphi_t)|\, f \circ \varphi_{-t}$$

- Cubical discretization, boxes B_i with equal measure $\mu(B_i)$

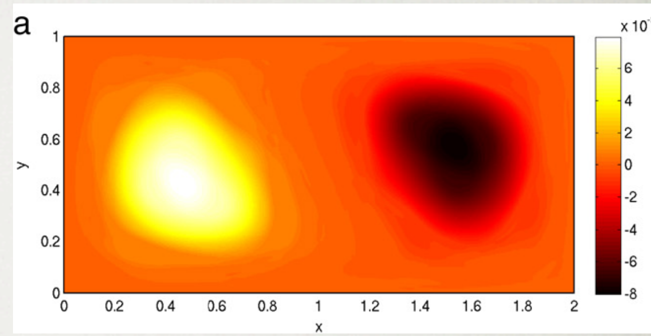
$$\mathcal{P}_{ij} = \frac{\mu(B_i \cap \varphi_{-t}(B_j))}{\mu(B_j)}$$

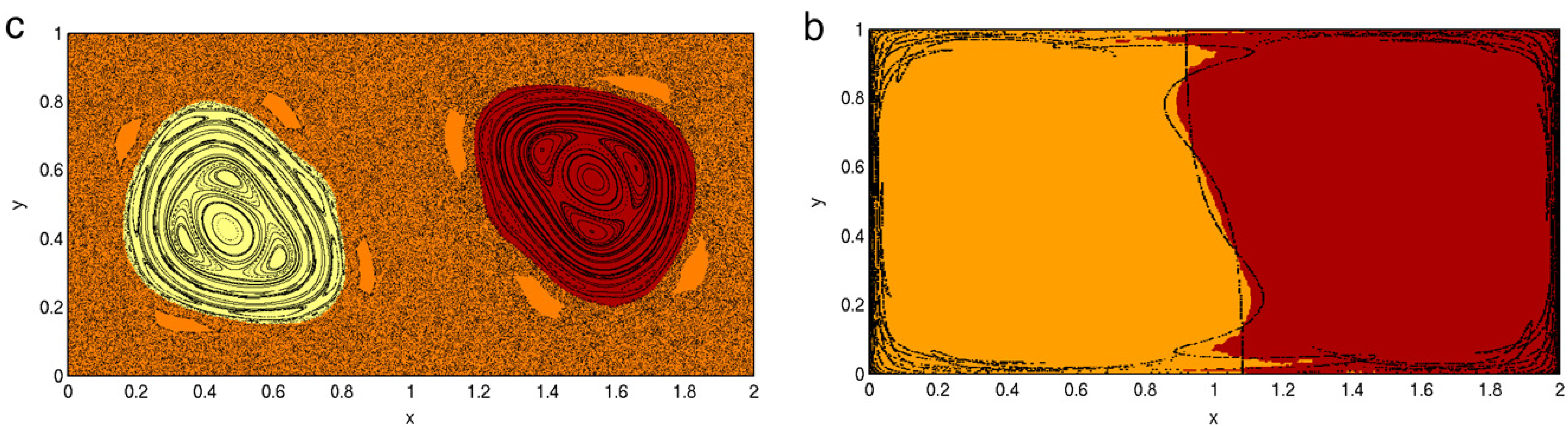
- Eigenvalue $\lambda = 1$
  - Measure preserving— uniform density
- Second isolated eigenvalue $\lambda < 1$
  - eigenfunction v partition into almost invariant sets $A^{\pm}$, $v_i > c$ and $v_i < c$.
  - Choose c to maximize *almost invariance,* $\varrho_t(A^{\pm})$

Dellnitz, M. and O. Junge (1999). "On the approximation of complicated dynamical behaviour." SIAM J. Numer. Anal. **36**(2): 491-451.
Froyland, G. and K. Padberg (2009). "Almost-invariant sets and invariant manifolds — Connecting probabilistic and geometric descriptions of coherent structures in flows." Physica D **238**: 1507-1523.

Double Gyre Example

- $2^{17} = 1.3(10)^5$ Boxes, 400 points/box
- Integration time = one period





Time one map $\qquad\qquad$ $c = 0$ partition

- Escaping flux for eigenfunction partition ($F = 0.0097$), is smaller than that through manifolds ($F = 0.0010$)

Froyland, G. and K. Padberg (2009). Physica D **238**: 1507-1523.

Questions

- Why are the eigenfunctions of the transition operator "better" than simply flowing a set forward?
  - Presumably the pair of coherent sets is more "regular" this way?
  - Does this regularity depend upon discretization?
  - Could some topological techniques like the entangled loop methods of J-L Thiffeault apply?

Thiffeault, J.-L. (2010). "Braids of entangled particle trajectories." Chaos **20**: 017516.

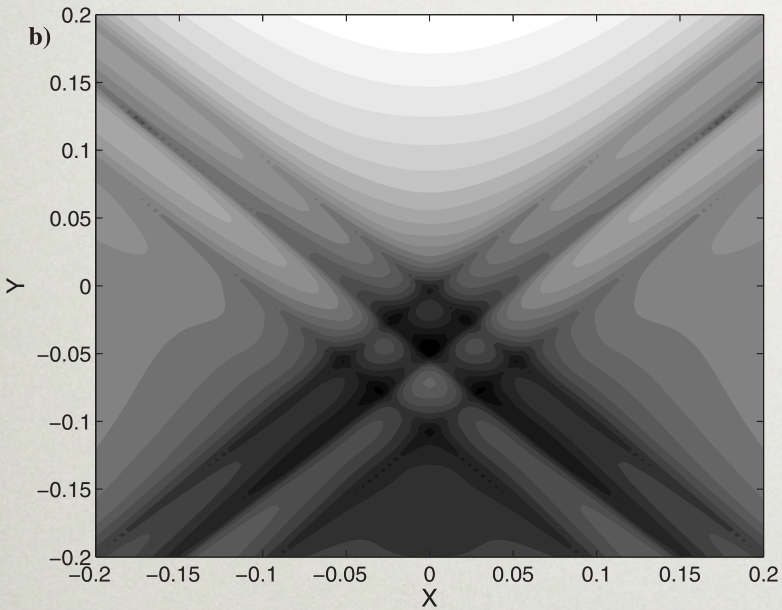
- Is the escaping flux from approximate invariant sets really lower than that for invariant manifolds (or cantori)?
- How are these related to the time average partitions of Mezic et al?

Distinguished Trajectories

Distinguished Trajectories

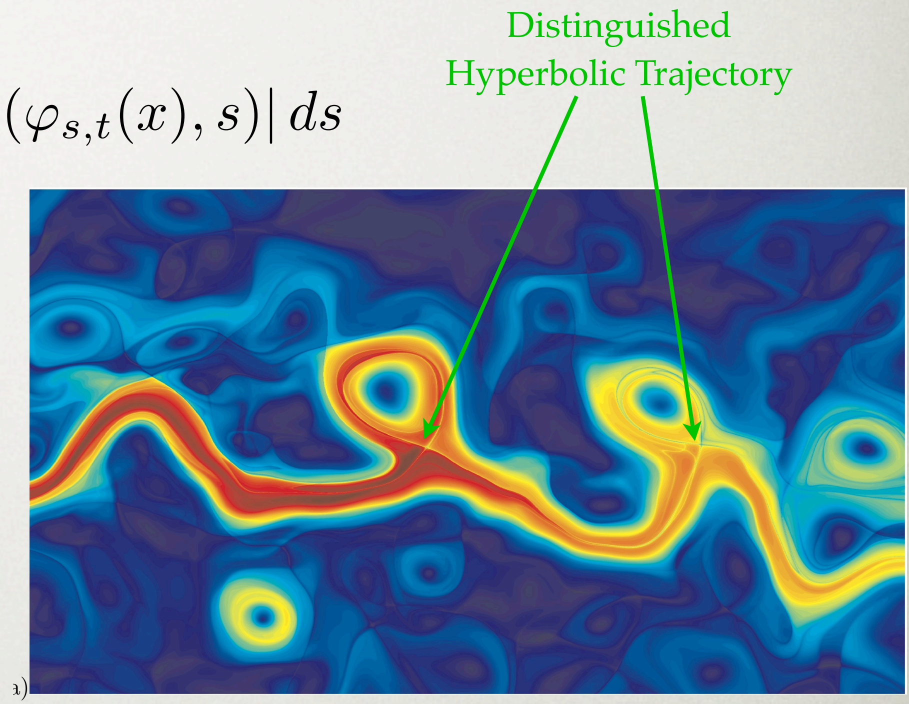
Jiménez Madrid & Mancho (2010): Local minima of a *Lagrangian descriptor*, e.g., "trajectory arc length"

$$M(x,t,\tau) = \int_{t-\tau}^{t+\tau} |V(\varphi_{s,t}(x), s)|\, ds$$



M(*x*,0,10) for the Forced Duffing oscillator

$$\ddot{x} = x - x^3 + 0.1\sin(t)$$

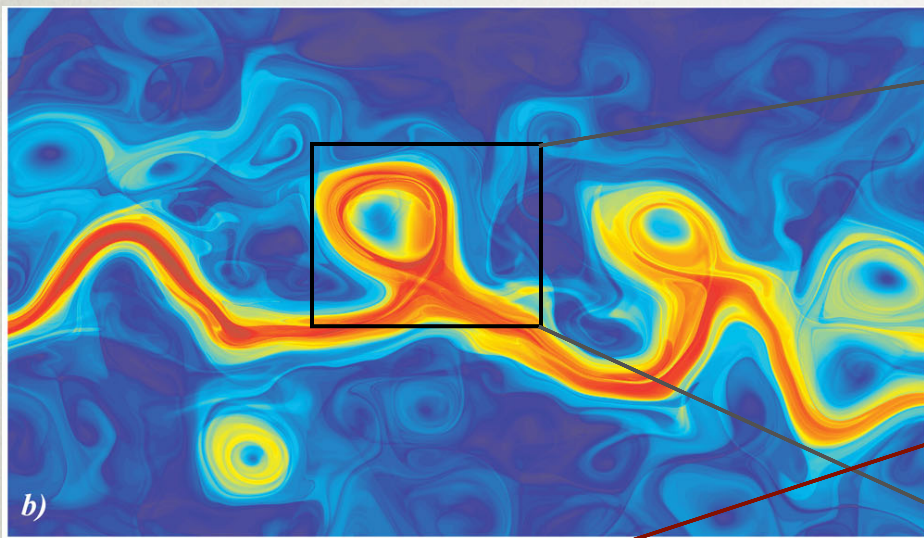


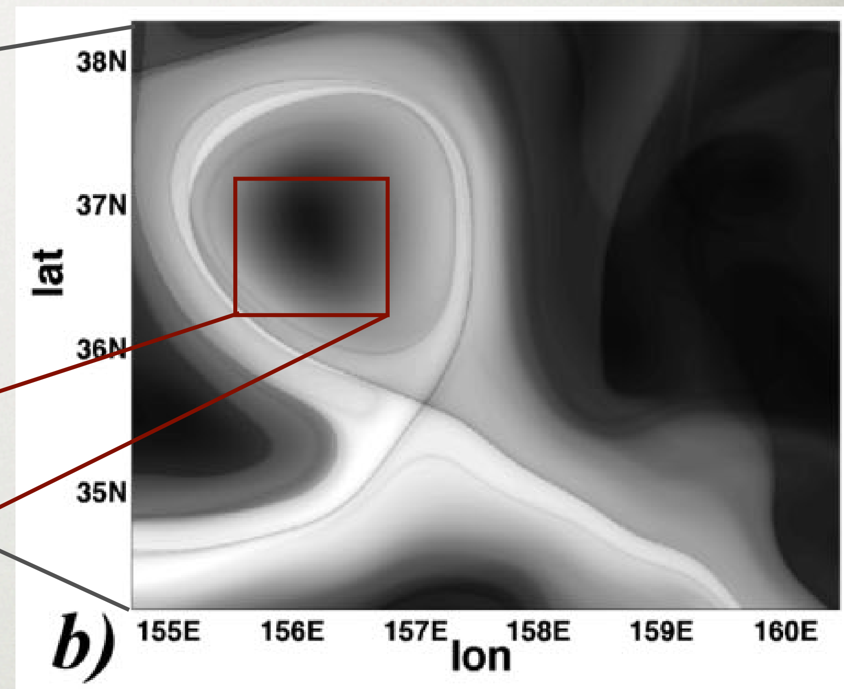
Kuroshio Current measurements 5/2003
Mendoza & Mancho (2010)

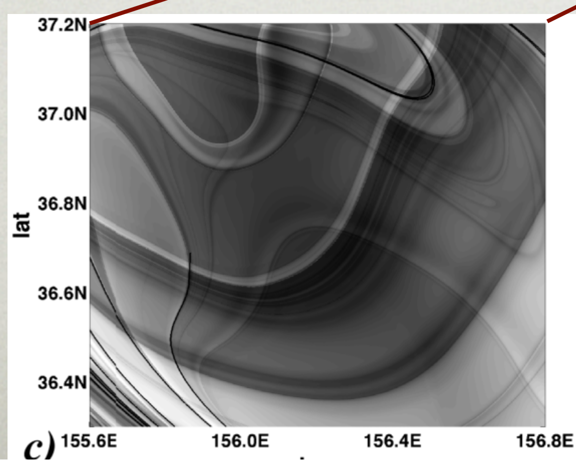
Lagrangian Descriptors

$$M(x,t,\tau) = \int_{t-\tau}^{t+\tau} |V(\varphi_{s,t}(x), s)|\, ds$$

- Sharp changes in *M* detect both stable and unstable







Kuroshio Current Data
May 2, 2003

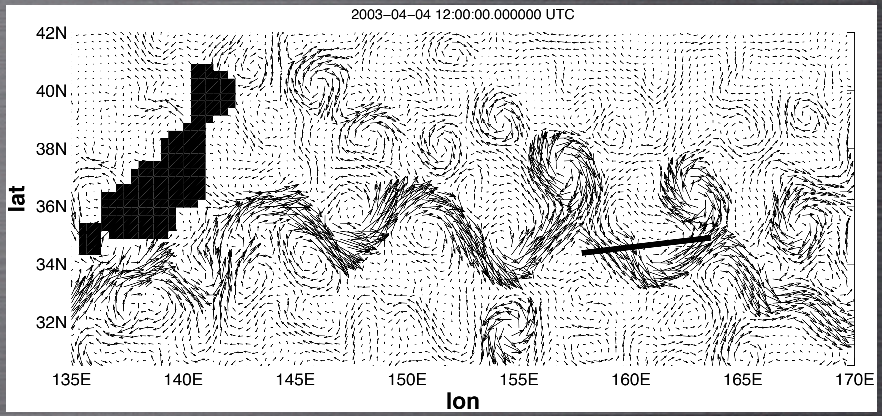
Mendoza, C. and A. M. Mancho (2010). "Hidden Geometry of Ocean Flows." Phys. Rev. Lett. **105**: 038501.

Questions

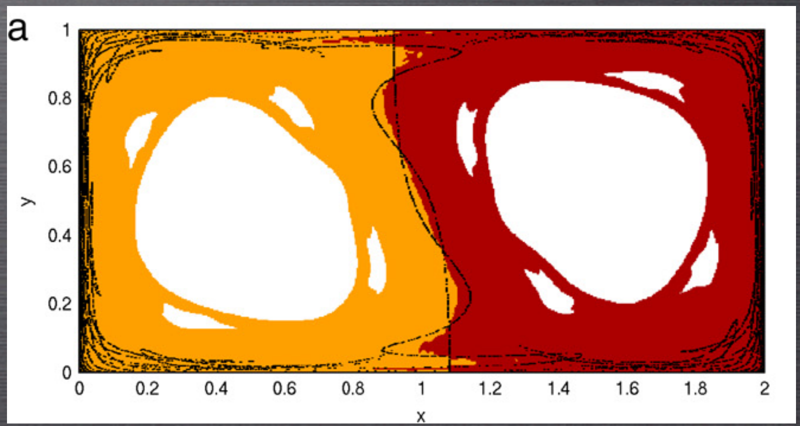
- How to pick time-scale? (Just as for FTLE)
- Is there a coordinate free Lagrangian descriptor
  - M depends upon reference frame?
- Does M detect hyperbolic periodic orbits or just fixed points?
- What about forward/backward descriptors?

Coming Attractions:

The Lagrangian Description of Aperiodic Flows: New Concepts and Tools
Ana Mancho



Set-oriented Numerical Analysis of Time-dependent Transport
Kathrin Padberg-Gehle



Lagrangian Transport Phenomena in 3D Laminar Mixing Flows
Michel Speetjens