NL3238 **The Standard Map**

Derivation

The standard or Taylor-Chirikov map is a family of area-preserving maps, $z' = f(z)$ with $z = (x, y)$, given by

$$\begin{aligned} x' &= x + y - \frac{k}{2\pi}\sin(2\pi x) \\ y' &= y - \frac{k}{2\pi}\sin(2\pi x) \ . \end{aligned} \tag{1}$$

Here x is a periodic configuration variable, and is usually computed "mod 1," and $y \in \mathbb{R}$ is the momentum variable. The map has a single parameter k which represents the strength of the nonlinear kick. This map was first proposed by Bryan Taylor and then independently obtained by Boris Chirikov to describe the dynamics of magnetic field lines. The standard map and Hénon's area-preserving quadratic map provide extensively studied paradigms for chaotic Hamiltonian dynamics.

The standard map is an *exact symplectic* map of the cylinder. Since $x'(x, y)$ is a monotone function of y for each x, it is also an example of a monotone twist map (see *Aubry–Mather theory*). Every twist map has a Lagrangian generating function; the standard map is generated by $F(x, x') = \frac{1}{2}(x' - x)^2 + \frac{k}{4\pi^2}\cos(2\pi x)$, so that $y = -\partial F/\partial x$ and $y' = \partial F/\partial x'$. The map can also be obtained from a discrete Lagrangian variational principle: define the discrete action for any configuration sequence $\ldots, x_{t-1}, x_t, x_{t+1}, \ldots$ as the formal sum

$$A[\ldots, x_{t-1}, x_t, x_{t+1}, \ldots] = \sum_t F(x_t, x_{t+1}) \ . \tag{2}$$

Then an orbit is a sequence which is a critical point of A; this gives the discrete Euler-Lagrange equation

$$x_{t+1} - 2x_t + x_{t-1} = -\frac{k}{2\pi}\sin(2\pi x_t) \ . \tag{3}$$

This second difference equation is equivalent to (1) upon defining $y_t = x_t - x_{t-1}$.

The standard map is an exact or approximate description of many physical systems. One example is the "kicked rotor:" consider a rigid body with moment of inertia I that is free to rotate in a horizontal plane about its center of mass. Suppose that an impulsive torque $\Gamma(\theta) = -A\sin(\theta)$ is applied to the rotor at times nT, $n \in \mathbb{Z}$. Let (θ_j, L_j) be angular position and angular momentum at time $jT - \epsilon$ for $\epsilon \to 0^+$. At time T later these become $(\theta_{j+1}, L_{j+1}) = (\theta_j + \frac{T}{I}L_{j+1}, L_j + \Gamma(\theta_j))$. Scaling these variables appropriately gives (1).

The standard map also describes the relativistic cyclotron, and is the equilibrium condition for a chain of masses connected by harmonic springs in a periodic potential—the Frenkel-Kontorova model. Similar maps include Chirikov's separatrix map (valid near the separatrix of a resonance) the Kepler map (describing the motion of comets under the influence of Jupiter as well as a classical hydrogen atom in a microwave field), and the Fermi map (for a ball bouncing between oscillating walls) Lichtenberg and Lieberman (1992). The higher dimensional version is the Froeshlé map (see *Symplectic maps*).

Symmetries

The standard map f has a number of symmetries which lead to special dynamical behavior. To see these, it is convenient to *lift* the map from the cylinder to the plane by extending the angle variable x to $\mathbb{R}$ (see *Circle maps*.

Let $T_{m,n}(x, y) = (x + m, y + n)$ be the translation by an integer vector (m, n). Since f is periodic, its lift has a discrete translation symmetry $f \circ T_{m,0} = T_{m,0} \circ f$. More unusually, the standard map also has a discrete vertical translation symmetry $f \circ T_{0,n} = T_{n,n} \circ f(x, y)$. Identifying orbits equivalent under these symmetries implies that standard map can be thought of as acting on the torus $\mathcal{T} = \{-\frac{1}{2} \leq x, y < \frac{1}{2}\}$.

The standard map also commutes with the reflection $S(x, y) = (-x, -y)$. This can be used to identify the lower half plane with the upper one, and to restrict the map to the space $\mathcal{S} = \{(x, y) : -\frac{1}{2} \leq x < \frac{1}{2},\ 0 \leq y \leq \frac{1}{2}\}$ identifying $(-\frac{1}{2}, y) \equiv (\frac{1}{2}, y)$ and each half of the upper and lower boundaries: $(x, 0) \equiv (-x, 0)$, $(x, \frac{1}{2}) \equiv (-x, \frac{1}{2})$. The map on the two sphere $\mathcal{S}$ is singular at the corners $(\pm\frac{1}{2}, 0)$ and $(\pm\frac{1}{2}, \frac{1}{2})$.

The standard map is also reversible: it is conjugate to its inverse $RfR^{-1} = f^{-1}$ (Lamb and Roberts, 1998). One reversor is $R_1(x, y) = (-x, y - \frac{k}{2\pi}\sin(2\pi x))$; this generates a family of reversors $R = f^n \circ R_1$. These reversors are involutions, $R^2 = id$, thus f can be written as the composition of two involutions $f = (f \circ R) \circ R$. Finally, the composition of a symmetry and a reversor is also a reversor, so that, for example $R_2 = SR$ is also a reversor.

Symmetric orbits are invariant under a symmetry or a reversor. This is particularly interesting since symmetric orbits must have points on the fixed sets of the reversor, $\mathrm{Fix}(R) = \{z : z = R(z)\}$ or on $\mathrm{Fix}(fR)$. Since these fixed sets are curves, symmetric orbits particularly easy to find. Rimmer showed that the bifurcations of symmetric orbits are special; for example, they undergo pitchfork bifurcations (see *Bifurcations*.

Dynamics

When $k = 0$, the dynamics of the standard map is integrable: the momentum y is an invariant. On each invariant circle $\mathcal{C}^0_\omega = \{(x, y) : y = \omega\}$, the angle after t iterates is given by $x_t = x_0 + \omega t \bmod 1$, thus the dynamics is that of the constant rotation, $R_\omega(\theta) = \theta + \omega$, on the circle with rotation number ω. When ω is rational every orbit on $\mathcal{C}^0_\omega$ is periodic; otherwise they are quasiperiodic and densely cover the circle.

When $|k| << 1$, Moser's version of the KAM theorem implies that most of these invariant circles persist; that is, there is a rotational invariant circle $\mathcal{C}_\omega$ on which the dynamics is conjugate to the rotation R_ω (see *Hamiltonian dynamics*). KAM theory applies to circles with Diophantine rotation number, i.e., $\omega \in \{\Omega : |n\Omega - m| > \frac{c}{n^\tau}\ \forall m, n \in \mathbb{Z}\ , n \neq 0\}$ for some $\tau \geq 1$ and $c > 0$. This excludes, of course, all of the rational rotation numbers as well as intervals about each rational, but still leaves a positive measure set. While it is difficult to obtain reasonable estimates for the interval of k for which all Diophantine circles (with given c and τ) persist, in 1985 Herman showed analytically that there is at least one invariant circle when $|k| \leq 0.029$, and de la Llave and Rana (1990) used a computer assisted proof to extend this result up to 0.91.

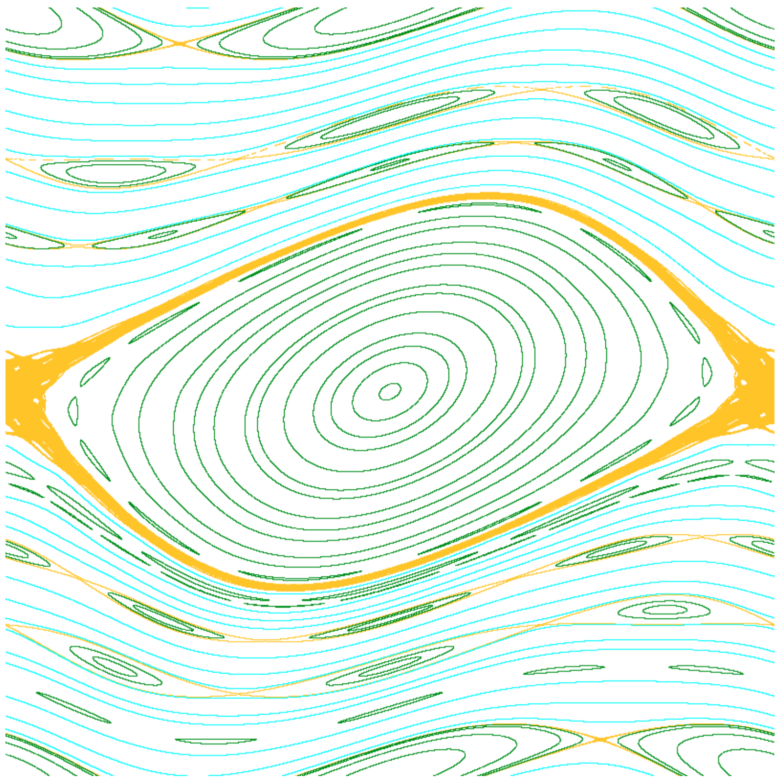

Figure 1. Dynamics of the standard map for $k = 0.6$ on the torus $\mathcal{T}$. Each of the blue curves is formed from many iterates on a rotational invariant circle like those predicted by the KAM theorem. The green orbits are secondary and tertiary circles arising from resonances. The gold orbits are chaotic trajectories near the stable and unstable manifolds of the resonances.

Some of the periodic orbits on the rational circles $\mathcal{C}^0_{m/n}$ also persist for nonzero k. Indeed the Poincaré-Birkhoff theorem implies that there are at least two period n orbits (with positive and negative Poincaré indices, respectively). *Aubry–Mather theory* implies that orbits with rotation number m/n can be found variationally; one is a global minimum of the action (2), and the other is a minimax point (a saddle of A with one downward direction). For example when $k > 0$, $(\frac{1}{2}, 0)$ is a minimizing fixed point, and $(0, 0)$ is a minimax fixed point. The reversibility of the standard map implies that there must be symmetric periodic orbits for each $\omega = m/n$ as well. Indeed it is observed that the minimax periodic orbits always have a point on the line $\mathrm{Fix}(R) = \{y = 0\}$, the "dominant" symmetry line.

The minimax orbits are elliptic when k is small enough. A convenient measure of stability of a period-n orbit is Greene's residue

$$R = \frac{1}{4}(2 - Tr(M)) \quad , \quad M = \prod_{t=0}^{n-1} Df(z_t) \; .$$

An orbit is elliptic when $0 < R < 1$. For example, the fixed point $(0, 0)$ has residue $k/4$. Perturbation theory shows that the residues of the minimizing and minimax orbits are $\mathcal{O}(k^n)$.

Each nondegenerate minimum of the action (2) is a hyperbolic orbit and has unstable and stable manifolds. For each minimizing m/n orbit, these intersect and enclose the minimax orbit, forming an island chain or resonance. A number of these chains are visible in Fig. [1]. The intersection of the manifolds is transverse, though the angle between them is exponentially small in k (Gelfreich and Lazutkin, 2001).

When stable and unstable manifolds intersect transversely, some iterate of the map has a *Smale horseshoe*. This implies that there is, at least, a cantor set of chaotic orbits. Umberger and Farmer (1985) showed numerically that there is a fat fractal set on which the dynamics has a positive Lyapunov exponent. The proof of this statement is still illusive. The regions occupied by chaotic orbits appear to grow in measure as k increases. Numerically it appears that a single initial condition densely covers each "zone of instability" a chaotic zone bounded by invariant circles, see Fig. [2]. There are also many elliptic periodic orbits that are created for nonzero k. For example, the $(0, 0)$ fixed point undergoes a period doubling bifurcation at $k = 4$ creating a period two orbit. More generally when the eigenvalues of any elliptic period-n orbit are $\lambda_\pm = e^{\pm 2\pi i \omega}$ then new orbits are born that encircle the original orbit and have relative rotation number ω. When $\omega = m'/n'$, these correspond to a chain of nn' islands. As Birkhoff realized, the newly created elliptic orbit also will undergo similar bifurcations, so that the phase space shows a structure of islands-around-islands, ad infinitum. This structure can even exhibit self-similarity (Meiss, 1992) just like the *Feigenbaum period doubling* sequence for dissipative systems.

The last invariant circle

In 1968 John Greene began studying the destruction of invariant circles in the standard map. He showed that sequences of periodic orbits, namely the minimizing and minimax m/n orbits, whose rotation numbers converge on a given irrational, can be used to determine the existence of a circle with that frequency. Suppose that ω has a continued fraction expansion $[a_0, a_1, \ldots]$, $a_j \in \mathbb{Z}^+$, and let $\frac{m_j}{n_j} = [a_0, a_1, \ldots, a_j]$ be the j^{th} convergent of ω. Greene conjectured that when the residues of these orbits $R_j \to 0$, as $n \to \infty$ then the invariant circle $\mathcal{C}_\omega$ exists— MacKay (1992) gave a proof of much of this.

For the standard map it appears that each rotational invariant circle exists only up to a critical value, $k = k_{cr}(\omega)$; this graph was called the "fractal diagram" by Schmidt and Bialek in 1982. The critical k vanishes at every rational and appears to have local maxima for each *noble* irrational ω. Percival called a number noble if its continued

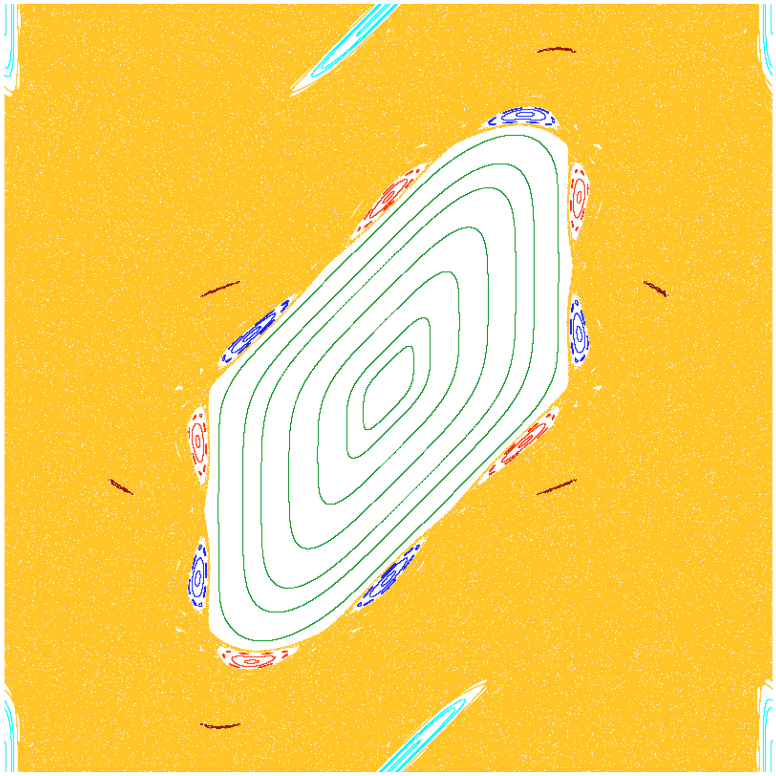

Figure 2. Phase space of the standard map for $k = 2.0$. At this value of k there are no rotational invariant circles. The gold region is filled by a single trajectory with $1.5(10)^6$ iterates. It appears to densely cover most of phase space, though there are still a number of secondary and tertiary islands visible. The red and blue island chains encircle the fixed point with rotation number 1/5; that there are two such chains is due to the reflection symmetry S.

fraction expansion has a tail that is eventually all ones. By this criterion the "most irrational" number is the golden mean $\gamma = \frac{1+\sqrt{5}}{2} = [1, 1, 1, \ldots]$. Indeed for the standard map, Greene discovered that the invariant circles with rotation numbers $\gamma \pm m$, $m \in \mathbb{Z}$ appear to be the last circles destroyed (all such circles are destroyed simultaneously due the symmetries). Numerically it is known that the golden circle is destroyed at $k_{cr}(\gamma) \approx 0.971635406$.

This value is most efficiently computed by renormalization theory (MacKay, 1993). At the critical parameter for the destruction of a noble invariant circle the phase space exhibits a self-similar structure, see Fig. [3]. The geometric scaling of this self-similarity can be used to compute k_{cr} from the residues of the m_j/n_j orbits. This is more accurate than iteration methods–pioneered by Chirikov–which rely on finding an orbit that crosses the region containing the circle, and frequency methods—developed by Laskar—which rely on the irregularity of the numerically computed rotation number. While none of these methods prove that k_{cr} corresponds to the last invariant circle, "converse KAM theory " leads to a computer proof that there are no rotational circles for $k > 63/64$ (MacKay and Percival, 1985). This is based on Birkhoff's theorem that every rotational invariant circle is a Lipschitz graph (Meiss, 1992).

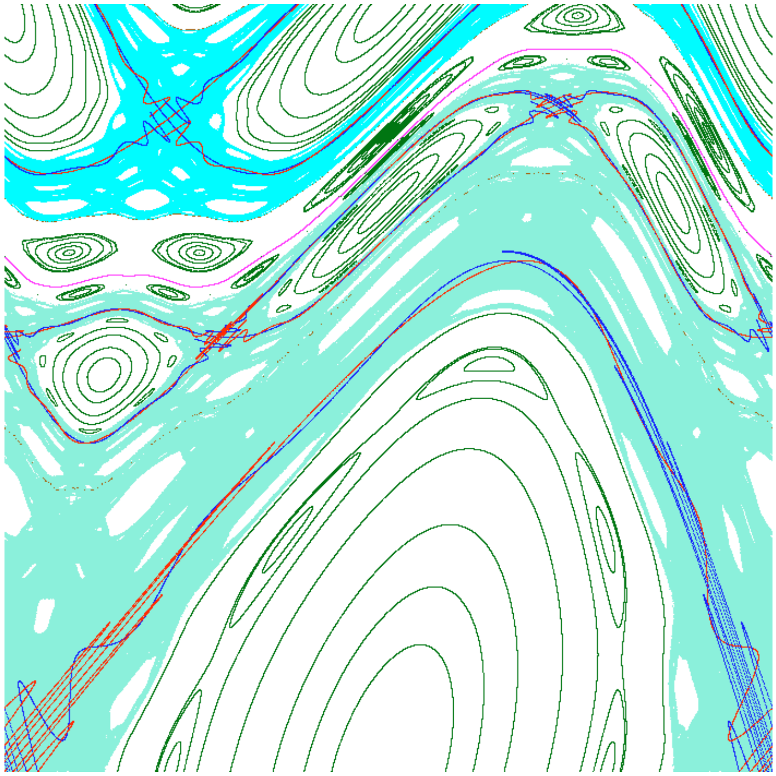

Figure 3. Dynamics of the standard map on the sphere $\mathcal{S}$ for $k = 0.971635406$, where the golden circle (purple) is critical. Also shown are $1.5(10)^6$ iterates of two chaotic trajectories (light blue and light green), the stable (blue) and unstable (red) manifolds of the $(m,n) = (0,1)$, $(1,2)$, and (1.3) orbits, and a number of orbits trapped in these island chains as well as the $(2,5)$ and $(3,8)$ chains. Finally there are two noble cantori shown (brown), with rotation numbers $(1+\gamma)/(3+4\gamma)$ and $(1+2\gamma)/(2+5\gamma)$

Transport

Transport theory studies the motion of ensembles of trajectories from one region of phase space to another. When there are invariant circles separating the regions, then there is no transport. A Birkhoff "zone of instability" is an annular region bounded by, but otherwise not containing any, rotational invariant circles. Birkhoff showed there are orbits that traverse each zone of instability, and Mather (1991) extended this to show that there are orbits future and past asymptotic to the upper and lower bounding rotational invariant circles, respectively.

Aubry–Mather theory implies that for each irrational rotation number there is a minimizing trajectory that is dense on a circle or a cantor set. Percival proposed calling the latter sets "cantori." Thus for $k > 63/64$, every rotational invariant circle has become a cantorus, and vertical transport between any two momentum levels occurs. The rate of transport is locally governed by the flux, the area that crosses a closed loop upon iteration. The flux across a cantorus or a separatrix is given by Mather's ΔW, the difference in action between the corresponding minimax and minimizing orbits (MacKay et al., 1984). Renormalization theory shows that the flux through a noble cantorus goes to zero as $(k - k_{cr})^{3.01}$; this can be very small well beyond k_{cr}. For example in Fig. [3], the blue chaotic trajectory is bounded below by a low flux cantorus even for tens of millions of iterates. Geometrically the flux is the area contained in a "lobe" bounded by pieces of stable and unstable manifolds; all transport occurs through lobes in two dimensional maps (Wiggins, 1992); unfortunately the higher-dimensional generalization is not clear.

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See also Aubry–Mather theory; Cat maps; Chaotic Dynamics; Circle maps; Ergodic theory; Fermi Map; Hénon map; Hamiltonian dynamics; Horseshoes and hyperbolicity; Lyapunov exponents; Maps; Measures; Melnikov method; Phase space; Symplectic maps.

Further Reading

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