Drift by Coupling to an Anti-Integrable Limit

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August 15, 2000
Revised April 9, 2001

Abstract

A symplectic twist map near an anti-integrable limit is conjugate to a full shift on a set of symbols. We couple such a system to another twist map in such a way that the resulting system is symplectic. At the anti-integrable limit we construct a set of nonzero measure of orbits of the second map that drifts arbitrarily far, even when the coupling is arbitrarily small. Moreover, these drifting orbits persist near the anti-integrable limit.

AMS classification scheme numbers: 37E40, 37J40, 37C50

1 Introduction

Are elliptic fixed points of symplectic maps generally stable or unstable? This question continues to prove an enticing problem. The answer was provided

*RWE would like to thank Rick Moeckel for inspiring some of the ideas in this paper during his lecture at the Celestial Mechanics conference in honor of Don Saari, Dec. 1999. We would also like to thank the referee and Robert MacKay for pointing out the argument based on Birkhoff's theorem in §4. JDM was supported in part by NSF grant number DMS-9971760. GR was supported by NSF Vigre grant number DMS-9810751

for the planar case by the celebrated work of Kolmogorov, Arnold and Moser (KAM): if the map satisfies a "twist condition" in a neighborhood of a fixed point and is not low-order resonant, then the point is encircled by a family of invariant curves and hence is stable [Arn78]. For $2n$ dimensions, KAM theory implies that when a twist condition is satisfied there exists a positive measure family of invariant n dimensional tori having the fixed point as a limit point. When $n > 1$, this family of tori does not necessarily prevent an orbit starting near the fixed point from wandering far away. The "density" of the family of tori should increase as the distance to the fixed point decreases, and the probability that the orbit of a randomly chosen point very close to the fixed point passes beyond a sphere of fixed radius about that point is close to zero. However, the fixed point, z, is by definition unstable if there exists a sequence of points $z_n \to z$, and a neighborhood U of z such that the orbit of each point z_n intersects the complement of U. It is possible that such a sequence exists even when the orbit of a randomly chosen point near z is likely to remain in U.

Arnold gave a famous example of this behavior in 1964 [Arn64]. Specifically he showed that a nearly-integrable hamiltonian flow with more than two degrees of freedom can have orbits that move arbitrarily far from their initial action values. This phenomena has come to be known as Arnold diffusion. The, as yet unsolved, "problem of Arnold diffusion" is to show that such a topological instability "typically" occurs in nearly-integrable hamiltonian systems (and perhaps eventually to show that the action drift is indeed diffusive). Arnold's example is special in that it is a periodically time dependent system with two degrees of freedom, constructed in such a way as to have a normally hyperbolic family of invariant two tori.

In this paper we are concerned with similar phenomena in symplectic maps. A symplectic analogue of Arnold's example is the map on $\mathbb{T}^2 \times \mathbb{R}^2$ defined by

$$\begin{aligned}
x' &= x + y' \ , \\
\xi' &= \xi + \eta' \ , \\
y' &= y - k(1 + h\cos\xi)\sin x \ , \\
\eta' &= \eta - kh(1 + \cos x)\sin\xi \ .
\end{aligned} \tag{1}$$

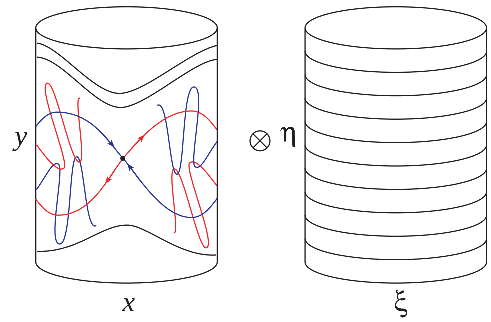

Figure 1: Phase space of the map (1) when $h = 0$ consists of two uncoupled maps. On the left is a standard map in (x, y), and on the right is an integrable twist map in (ξ, η).

This is a symplectic map preserving the two form $\omega = dx \wedge dy + d\xi \wedge d\eta$. When $h = 0$, the map (1) reduces to a pair of uncoupled, area-preserving maps, see Fig. 1. The (ξ, η) dynamics are integrable: each circle $\eta = a$ is invariant. The (x, y) dynamics are given by a "standard map," which is not integrable.[1] When $k > 0$ there is a cylinder $\mathcal{C} = \cup_a C_a$ foliated by the one parameter family of hyperbolic invariant circles $C_a = \{x = \pi, y = 0, \eta = a\}$. These circles have two dimensional stable and unstable manifolds and are called "whiskered tori." These manifolds intersect transversely giving a family of homoclinic orbits to each C_a. The map is specially chosen so that the term $1 + \cos x$ in the η component vanishes on these circles; this implies that each circle, C_a, exists for all h. Moreover, because the cylinder is normally hyperbolic at $h = 0$, the cylinder itself persists as a normally hyperbolic invariant set. The problem addressed by Arnold is to show that there are orbits that drift arbitrarily far in η when h is arbitrarily small.

For his example, Arnold constructs a "transition chain" of whiskered tori. For (1), this requires that one shows that the unstable manifold of an invariant circle C_a intersects transversely with the stable manifold of a nearby circle $C_{a'}$ whenever a' is close enough to a, see Fig. 2. Once the transition chain is obtained, the final step is to argue that orbits beginning near C_a can

[1] Thus this example differs from Arnold's in that he considers perturbations of an integrable system. This could be easily remedied by replacing the standard map with a Suris map [Sur89]. This map has a homoclinic connection from the fixed point $(\pi, 0)$ to itself, and perturbations can be studied using Melnikov theory [LM00].

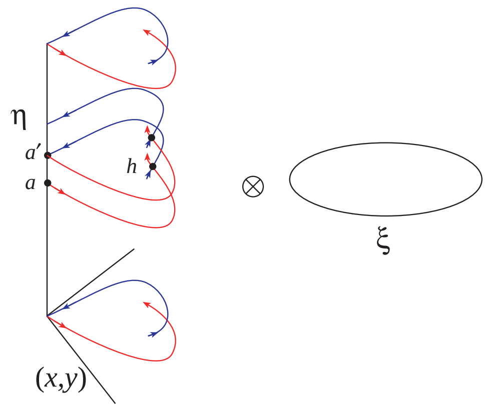

Figure 2: Phase space of the map (1) when $h \neq 0$. On the left we sketch the dynamics near the normally hyperbolic cylinder $\mathcal{C}$ suppressing the angle ξ, and showing the orbits homoclinic to the invariant circles C_a. The transition from C_a to $C_{a'}$ occurs at the heteroclinic point h.

drift near to $C_{a'}$ by shadowing the heteroclinic orbits. In this way one can construct orbits that move an arbitrary distance in η for h arbitrarily small.

Both Arnold's flow and the map (1) are special for two reasons. First since we assume $k > 0$ is independent of h, the cylinder $\mathcal{C}$ is a normally hyperbolic invariant manifold. The case when k and h are linked is important for the determination of the stability of a generic elliptic fixed point; however, it is difficult [Loc99]. The second distinguishing characteristic is that all of the whiskered tori persist when the coupling is nonzero. More generally one needs to use KAM theory to show that some subset of the tori (with Diophantine frequencies) persist. Unfortunately this means that there are gaps between the tori so that a' cannot be chosen arbitrarily closely to a. In this case one would need to show that the heteroclinic connections can bridge the gaps. In general this is also a difficult problem, see [Loc99] for a review. A variational approach pioneered by Mather can successfully obtain some results along these lines [BT99, DdlLS00].

We will avoid these difficulties by approaching the problem from a different direction. When $h = 0$ and k is large enough, there are many zero measure sets on which the standard map dynamics is hyperbolic. For the purposes of this introduction suppose that there is a hyperbolic Cantor set

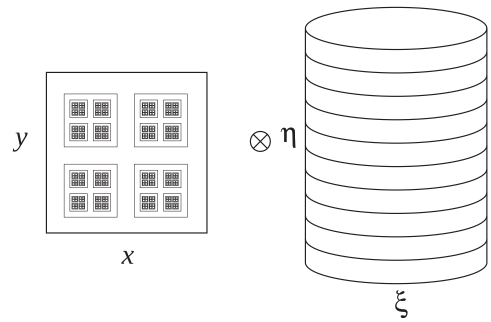

Figure 3: Phase space of an uncoupled map corresponding to a Cantor set cross an integrable twist map.

in (x, y) that is conjugate to a full shift on a set of symbols. Instead of considering the basic system to be the cylinder $\mathcal{C}$, we focus on the Cantor set cross the cylinder (ξ, η), see Fig. 3. The idea is to choose an orbit in the Cantor set that "drives" the momentum η to drift when the coupling is nonzero. Essentially we will use the symbolic description of the Cantor set to decompose it into subsets for which the coupling drives η to increase or to decrease. Careful choice of the symbol sequence can give an orbit with the required behavior. Indeed, Moeckel has shown that the existence of drifting orbits is generic in a similar situation [Moe00].

To show that such orbits exist at zero coupling and persist when the coupling is nonzero, we utilize the "anti-integrable" limit of the standard map, see §3. In this limit, the requisite Cantor set is easy to obtain, and a straightforward application of the implicit function theorem can be used to show that orbits with bounded acceleration persist. We will see in §4 that when the coupling is nonzero, a nonzero measure of orbits in the cylinder (ξ, η) can be appropriately driven so that their momenta increase by an arbitrarily large amount. We call such orbits "drifting orbits." This argument is based on the construction of a map on the cylinder with nonzero net flux, see §5.

Since the dynamics on the cylinder are not hyperbolic, we cannot use a shadowing argument to show that this behavior persists away from the anti-integrable limit; however, a continuity argument, given in §6, implies that there are many orbits whose momenta drift arbitrarily far.

2 Twist Maps

We begin by recalling some of the basic facts about symplectic twist maps [Mei92]. We consider a map $(x', y') = f(x, y)$, where $x \in \mathbb{T}^n$ represent the configuration angles and $y \in \mathbb{R}^n$ the canonical momenta. We will often use the same symbol x to represent a point on the universal cover, $\mathbb{R}^n$, of $\mathbb{T}^n$. The map f is an exact, symplectic map when the one form $\alpha = y'dx' - ydx$ is exact, i.e., $\alpha = dL$ for some function L. We say the map has twist when the function L can be written as $L(x, x')$, depending upon the coordinates x and x'; in this case L is the discrete analogue of the Lagrangian. The map is then defined implicitly through

$$y'dx' - ydx = dL = L_1 dx + L_2 dx' \ ,$$

where $L_1(x, x') \equiv \partial L/\partial x$, and $L_2(x, x') \equiv \partial L/\partial x'$. These equations define the map f when the relation $y = -L_1(x, x')$ is a diffeomorphism for each fixed x. In this case the inverse of this relation is the projection of f onto the configuration components, $x' = \pi f(x, y)$. It is easy to see that the map is exact when the net flux

$$\mathcal{F}_m = L(x + 2\pi m, x' + 2\pi m) - L(x, x') \ , \tag{2}$$

vanishes for each pair (x, x') in the covering space, and each vector $m \in \mathbb{Z}^n$.

It is convenient to lift the configuration coordinates to $\mathbb{R}^n$ and to consider a sequence of configuration points $X = \{x_t : t \in \mathbb{Z}\}$. In this case, an orbit is a critical point of the formal action sum

$$W(X) = \sum_{t=-\infty}^{\infty} L(x_t, x_{t+1}) \ .$$

The condition that $DW = 0$ gives the discrete Euler-Lagrange equations

$$L_2(x_{t-1}, x_t) + L_1(x_t, x_{t+1}) = 0 \ .$$

It is common in applications for the generating function L to have the *natural* form

$$L(x, x') = \varepsilon T(x, x') + V(x) \ , \tag{3}$$

where T represents a discrete kinetic energy and V represents (minus) the potential energy. We insert the coefficient ε for later use. When $T = \frac{1}{2}(x' - x)^2$, then the Euler-Lagrange equations are equivalent to a *generalized standard map*

$$\varepsilon(x_{t+1} - 2x_t + x_{t-1}) = DV(x_t) ,$$

which is a discretized version of Newton's second law. In terms of the canonical variables $(x, y) \mapsto (x', y')$, the generalized standard map is

$$\begin{aligned} x' &= x + \frac{1}{\varepsilon} y' , \\ y' &= y + DV(x) . \end{aligned} \tag{4}$$

The case when the configuration space is one dimensional, $n = 1$, and $V(x) = k\cos(x)$, corresponds to *the* standard map [Mei92].

3 Anti-Integrable Limit

A discrete dynamical system is said to have an anti-integrable limit [Aub92, MM92, Aub95], when the dynamics reduces to a full shift on a discrete phase space. For example, the variational principle for the natural system (3) reduces to

$$\sum_t V(x_t) ,$$

when $\varepsilon = 0$. In this case the Euler-Lagrange equations reduce to $DV(x_t) = 0$, which implies that any sequence of critical points of the potential is an orbit. We restrict our consideration to those critical points that are (uniformly) nondegenerate. Thus the set of orbits is equivalent to a full shift on the set of uniformly nondegenerate critical points of V:

$$\mathrm{crit}(V) \equiv \{c : DV(c) = 0,\ ||D^2V(c)|| \geqslant b > 0\} . \tag{5}$$

Any sequence in $\mathrm{crit}(V)$ corresponds to an "orbit" when $\varepsilon = 0$. Of course, this limit is singular, in the sense that the map is no longer deterministic; moreover, since $L_2 = 0$, the momentum y' is not defined.

More generally, consider the Banach space $\mathcal{B} = (\mathbb{R}^n)^{\mathbb{Z}}$ of bi-infinite sequences equipped with the sup norm. For any $X \in \mathcal{B}$, define the acceleration X as

$$A_t(X) \equiv -T_2(x_{t-1}, x_t) - T_1(x_t, x_{t+1}) \ . \tag{6}$$

The reason for calling this the acceleration is that for generalized standard maps, A becomes the sequence of second differences of x_t: the discrete acceleration. We let $\mathcal{A} \subset \mathcal{B}$ be the subspace of sequences for which the maximal acceleration is a:

$$\mathcal{A} = \{X \in \mathcal{B} : ||A(X)|| \leq a\} \ . \tag{7}$$

When $\varepsilon \neq 0$ the Euler-Lagrange equation becomes

$$DV(x_t) = \varepsilon A_t(X) \ . \tag{8}$$

Note that a sequence $X \in \mathrm{crit}(V)^{\mathbb{Z}}$ is a solution of this equation at $\varepsilon = 0$, and that if $X \in \mathcal{A}$ the right hand side of this equation is bounded. Moreover, since the critical points are assumed to be uniformly nondegenerate, the left hand side has a uniformly bounded inverse. In this case, the implicit function theorem gives the following

Theorem 1. *([MM92]) For each sequence $X(0) \in crit(V)^{\mathbb{Z}} \cap \mathcal{A}$, there is an $\varepsilon_0(a, b) > 0$ such that there is a unique continuation of $X(0)$ to an orbit $X(\varepsilon)$ that satisfies (8) for all $|\varepsilon| < \varepsilon_0$.*

Here we intend to obtain related results for a system that is the direct product of an anti-integrable system with a symplectic map. The idea is to show that we can decompose the system into a semidirect product of a hyperbolic map with a symplectic map.

4 Coupled Systems

In this paper we wish to study maps that are coupled to a system near an anti-integrable limit. Denote the configuration by $(x, \xi) \in \mathbb{T}^m \times \mathbb{T}$, and consider a generating function of the form

$$L(x, \xi, x', \xi') = \varepsilon T(x, x') + T(\xi, \xi') + V(x)[1 + C(\xi)] + W(\xi) \ . \tag{9}$$

Here the periodic functions V and W represent the potential energies of separated systems, and the periodic function C represents the coupling. For example, the Arnold analogue (1) occurs when $W = 0$, $V = k(1 + \cos x)$, and $C = h\cos\xi$. Since V and W are periodic, L has zero net flux, (2), and therefore generates an exact, symplectic twist map, $f(x, \xi, y, \eta)$, on $\mathbb{T}^m \times \mathbb{T} \times \mathbb{R}^m \times \mathbb{R}$. Here the momenta are defined by $y = -\partial L/\partial x$ and $\eta = -\partial L/\partial \xi$.

This class of maps generalizes (1), but retains some special features. When C is identically zero, the dynamics becomes the direct product of two area-preserving twist maps. Neither of these maps is integrable. Moreover (9) does not generally have an invariant, hyperbolic cylinder. When $\epsilon = 0$ the map in (x, y) is at an "anti-integrable" limit. Our goal is to show that we can utilize the anti-integrable states to drive the momentum η to arbitrarily large values even when C is arbitrarily small.

The Euler-Lagrange equations for (9) have the form

$$\begin{aligned} \varepsilon A_t(X) &= DV(x_t)[1 + C(\xi_t)] \ , && (10) \\ A_t(\Xi) &= DW(\xi_t) + V(x_t)DC(\xi_t) \ , && (11) \end{aligned}$$

where $X = \{x_t\}$ and $\Xi = \{\xi_t\}$. These equations take a second difference form when $T(x, x') = \frac{1}{2}(x' - x)^2$:

$$\begin{aligned} \varepsilon(x_{t+1} - 2x_t + x_{t-1}) &= DV(x_t)[1 + C(\xi_t)] \ , \\ \xi_{t+1} - 2\xi_t + \xi_{t-1} &= DW(\xi_t) + V(x_t)DC(\xi_t) \ . \end{aligned} \tag{12}$$

Drift at the Anti-Integrable Limit

When $\varepsilon = 0$, (10) becomes particularly simple: $DV(x_t)(1+C(\xi_t)) = 0$. There are several possible solutions to this set of equations, however, we consider only the nondegenerate critical points of V:

$$x_t = c_t \in \mathrm{crit}(V) \ . \tag{13}$$

where $\mathrm{crit}(V)$ was defined in (5). We discard the other solutions because the degenerate critical points may not persist, and points where $C(\xi) = -1$ do not correspond to orbits in general.

When $\varepsilon = 0$ and we choose a particular sequence c_t in the critical set, (11) has the form

$$A_t(\Xi) = DW(\xi_t) + V(c_t)DC(\xi_t) \; , \tag{14}$$

which is a sequence of area-preserving maps. Orbits in this limit consist of solutions of (13), arbitrary sequences of critical points, together with solutions of (14).

We wish to show that the ξ dynamics can be driven far from its unperturbed motion even when C is arbitrarily small. For example, when $W = 0$, (14) implies that the acceleration is vanishingly small as $C \to 0$. Thus the momentum, $\eta_t = T_2(\xi_{t-1}, \xi_t)$, becomes constant. Our goal is to show that there are orbits that drift for arbitrarily small C. These can be constructed as follows. Suppose that $\mathrm{crit}(V)$ has nonempty subsets $\mathrm{crit}(V)_\pm$ such that if $c \in \mathrm{crit}(V)_+$, then $V(c) > 0$, and if $c \in \mathrm{crit}(V)_-$, then $V(c) < 0$. Then for each step we can choose c_t so that the map has a net flux.

Theorem 2. *Consider the Lagrangian (9) at $\epsilon = 0$ and let (x, ξ, y, η) denote the coordinates of the phase space $\mathbb{T}^m \times \mathbb{T} \times \mathbb{R}^m \times \mathbb{R}$. Assume that the set crit(V) has nonempty subsets crit$(V)_\pm$ where* $\mathrm{sign}(V) = \pm 1$, *and assume that DC is not identically zero. Then given any $a < b$, there is a nonzero measure of initial states (ξ_0, ξ_1) and a sequence $c_t \in$ crit$(V)_+ \cup$ crit$(V)_-$ such that the solution of (14) has momenta, $\eta_t = T_2(\xi_{t-1}, \xi_t)$ satisfying $\eta_0 < a$ and $\eta_T > b$ for some time T.*

Proof. Choose any two points $c_- \in \mathrm{crit}(V)_-$ and $c_+ \in \mathrm{crit}(V)_+$. For a point ξ_t, let $x_t = c_\pm$ if $\mathrm{sign}(DC(\xi_t)) = \pm 1$. In this case the map (14) can be thought of as an autonomous twist map, obtained from the generating function

$$\tilde{L}(\xi, \xi') = T(\xi, \xi') + W(\xi) + \tilde{C}(\xi) \; ,$$

where $\tilde{C}$ is the function whose derivative is

$$D\tilde{C} = V(c_\pm(\xi))DC(\xi) \geqslant 0 \; .$$

Thus $\tilde{C}(\xi+2\pi) - \tilde{C}(\xi) > 0$, and so the net flux $\mathcal{F} = \tilde{L}(\xi+2\pi, \xi'+2\pi) - \tilde{L}(\xi, \xi')$ is positive. As we will see below in Cor. 5, this implies there is a nonzero measure of drifting orbits. $\square$

Standard Example

Consider the four dimensional Lagrangian

$$L(x, x', \xi, \xi') = \frac{\varepsilon}{2}(x' - x)^2 + \frac{1}{2}(\xi' - \xi)^2 + k\cos x(1 + h\cos\xi) \ , \qquad (15)$$

where we take $k > 0$ and $h > 0$. Unlike the Arnold analogue (1), the hyperbolic invariant circles of (15) that exist for $h = 0$ at $x = \pi$ are not all preserved when the coupling is nonzero. Thus the Arnold argument does not directly apply to this system (one would have to deal with gaps between the circles). Nevertheless, we will see there is a set of drifting orbits at the anti-integrable limit and, from the results in §6, near it as well.

Since $V = k\cos x$ we have $\mathrm{crit}(V)_+ = \{2n\pi : n \in \mathbb{Z}\}$, and $\mathrm{crit}(V)_- = \{(2n+1)\pi : n \in \mathbb{Z}\}$. Since $kh > 0$, we choose $x_t \in \mathrm{crit}(V)_-$ for $0 \leqslant \xi_t < \pi$ and $x_t \in \mathrm{crit}(V)_+$ for $\pi \leqslant \xi_t < 2\pi$. In canonical coordinates, the effective map at $\varepsilon = 0$, (14), becomes

$$\begin{aligned} \xi_{t+1} &= \xi_t + \eta_{t+1} \ , \\ \eta_{t+1} &= \eta_t + U'(\xi_t) \ . \end{aligned} \qquad (16)$$

where $U'(x) = kh|\sin\xi_t|$. For this simplest case, the promised sequence of maps reduces to a single, piecewise smooth map. In fact this system is a twist map of the cylinder and is generated by the Lagrangian

$$\tilde{L}(\xi, \xi') = \frac{1}{2}(\xi' - \xi)^2 + U(\xi) \ ,$$

where the potential function is

$$U(\xi) = \begin{cases} kh(4n - \cos\xi) & 2n\pi < \xi \leq (2n+1)\pi \\ kh(4n + 2 + \cos\xi) & (2n+1)\pi < \xi \leq (2n+2)\pi \end{cases} .$$

The net flux is

$$\mathcal{F} = \tilde{L}(\xi + 2\pi, \xi' + 2\pi) - \tilde{L}(\xi, \xi') = 4kh \ ,$$

and so our general results of the next section imply there is a nonzero measure of drifting orbits.

In fact, the dynamics of this map are quite easy to understand: almost every orbit is unbounded. For (16) implies that the sequence η_t is monotone nondecreasing; thus if it is bounded it has a limit, η^*. For this to be the case, we must have ξ_t approaching 0 or π or some sequence of these points. There are three possibilities: if $\xi_t \to 0$ or $\xi_t \to \pi$ then $\eta^* = 2\pi m$. These points, $(0, 2\pi m)$ and $(\pi, 2\pi m)$, are the fixed points of (16). The only other possibility is that $\eta^* = (2m+1)\pi$, in which case the orbit approaches the period two orbit $(0, \eta^*) \mapsto (\pi, \eta^*)$.

The map (16) is area-preserving since it is generated by a Lagrangian (even though the map is not smooth it is the composition of a pair of shears, each of which is area-preserving). An orbit of a measure-preserving map attracts a set of at most measure zero, because if the stable set had nonzero measure it would eventually be mapped into a ball of arbitrarily small measure about the orbit, violating measure preservation. Thus the only bounded orbits of (16) are the three orbits that we found and the set of zero measure which limits on these. Therefore, almost all orbits are unbounded.

With an effective, area-preserving map of the form (16), an alternative argument based on the Birkhoff ergodic theorem, can also be used. Recall that this theorem states that the time average, g^* of a measurable function, g exists almost everywhere, and the space average of g^* is equal to the space average of g. The map (16) is periodic both in the coordinate and momentum directions, and so can be considered to be a map on the 2π torus $\mathbb{T}^2 = \{(\xi, \eta) : 0 < \xi, \eta < 2\pi\}$. We are interested in average drift rate of the momentum,

$$\Delta\eta^* = (\eta_{t+1} - \eta_t)^* = \lim_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} (U'(\xi_t))$$

Averaging this over all initial conditions, and using the Birkhoff theorem implies that

$$< \Delta\eta^* >= \frac{1}{4\pi^2} \int_{\mathbb{T}^2} U'(\xi) = \frac{\mathcal{F}}{2\pi} \; .$$

Whenever the spatial average of the average momentum change per iteration is positive, there must be a nonzero measure of orbits that is unbounded. Thus whenever the map (16) has nonzero net flux, there is a nonzero measure of unbounded orbits. An argument similar to this has been used to show that there are drifting orbits for other area-preserving maps as well as time periodic flows in [MBW96].

We will see in §5 that when the potential $W \neq 0$, the bounded set can have nonzero measure, but when the net flux is nonzero, there is always a set of positive measure that drifts.

5 Net Flux

Imagine a cafe so popular that it is, at all times, completely full. The entrance is carefully guarded and patrons are admitted only when others leave. The establishment has two exits, front and back. Patrons leave at a varying rate, which is the same as the rate at which new customers are admitted. Suppose that the front exit is always too narrow to accommodate all of those leaving. It is perhaps intuitively clear that some of the entering patrons must eventually leave through the back door. This is true even if the back door is almost inaccessible and there is a large, unyielding group who never leave.

Below we formalize this simple result. We then apply it to maps of the cylinder with nonzero net flux and show that orbits must drift. Here the cafe is replaced by an annulus, and patrons are collapsed to points.

Transitions

We begin by recalling some general definitions in transport theory [Eas91, Mei97]. For a map f and a region R, we define the incoming set as

$$I = \{z \in R : f^{-1}(z) \notin R\} = R \setminus f(R) \ ,$$

and the exit set as

$$E = \{z \in R : f(z) \notin R\} = R \setminus f^{-1}(R) \ .$$

When f is one-to-one and preserves a measure, μ, and R is measurable,

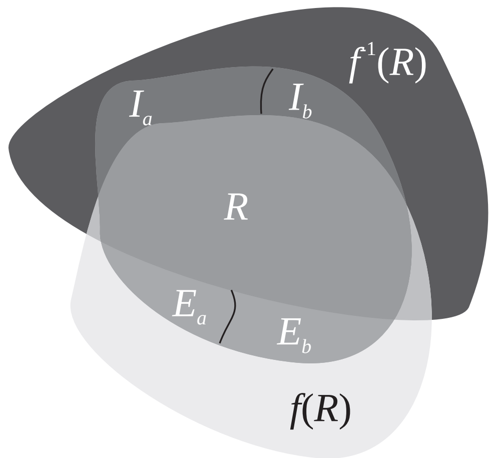

Figure 4: Incoming and exit sets for a region R.

then the measures of the incoming and exit sets must be equal:

$$\begin{aligned} \mu(E) &= \mu(R \setminus f^{-1}(R)) = \mu(R) - \mu(R \cap f^{-1}(R)) \\ &= \mu(R) - \mu(f(R) \cap R) = \mu(R \setminus f(R)) \\ &= \mu(I) \ . \end{aligned} \tag{17}$$

Moreover, almost all orbits that start in the incoming set must eventually fall in the exit set. To see this let S^t be the portion of the t^{th} image of I that *stays* in R for all times up to t. These sets can be defined recursively by

$$\begin{aligned} S^0 &= I \ , \\ S^t &= f(S^{t-1}) \cap R = f(S^{t-1} \setminus E) \ . \end{aligned}$$

The sets S^t are disjoint, since their preimages $f^{-j}(S^t)$ are in R for $j = 0 \ldots t$, but their $(t+1)^{st}$ preimage is not. Since the S^t are subsets of R we have $\sum_{t=0}^{\infty} \mu(S^t) < \mu(R)$. Thus $\mu(S^t) \to 0$ as $t \to \infty$. It follows that almost all points in I must eventually leave R and therefore must land at some time in E.

We can say even more:

Lemma 3. *Let f be a measure preserving homeomorphism, and R a measurable set with incoming set I and exit set E. Define disjoint decompositions*

$I = I_a \cup I_b$ and $E = E_a \cup E_b$, such that $\mu(I_a) > \mu(E_a)$, c.f., Fig. 4. Then the subset of orbits starting in I_a that first leaves R through E_b has a measure at least $\mu(I_a) - \mu(E_a)$.

Proof. Almost all points in I must eventually iterate to points in E. Let T^j be the transit time decomposition of E, i.e., the decomposition into sets $T^j = S^j \cap E$. The sets T^j are disjoint, and since their j^{th} preimages are subsets of I we can define the first transit Poincaré map, $p : I \to E$, as

$$p(x) = f^j(x) \text{ if } x \in f^{-j}(T^j) ,$$

for almost all points in I. Since f is measure preserving, so is p. The image of a portion of the incoming set can be decomposed as $p(I_a) = p(I_a) \cap E_a + p(I_a) \cap E_b$, and we note that $\mu(p(I_a) \cap E_a) \leq \mu(E_a)$. Thus the measure of orbits that leave through E_b is

$$\mu(p(I_a) \cap E_b) = \mu(p(I_a)) - \mu(p(I_a) \cap E_a) \geqslant \mu(I_a) - \mu(E_a) .$$

□

A similar result can be obtained for a sequence of maps f_t. For a fixed region R, the incoming and exit sets at time t are defined by

$$\begin{aligned} I_t &= R \setminus f_t(R) , \\ E_t &= R \setminus f_t^{-1}(R) . \end{aligned}$$

When f_t is measure preserving, the measures of these sets must be equal: we can transcribe the previous argument, (17), directly. We can similarly define the set which has just entered R at time k and stays to at least time t by S_k^t. These obey recursion relations similar to the previous ones

$$\begin{aligned} S_k^k &= I_{k-1} , \\ S_k^{t+1} &= f_t(S_k^t \setminus E_t) . \end{aligned}$$

As before the sets S_k^t for a fixed t and for all $k \leq t$ are disjoint subsets of R because they have distinct prehistories under the sequence of maps; thus

$$\sum_{k=-\infty}^{t} \mu(S_k^t) < \mu(R) , \tag{18}$$

for each t.

Lemma 4. *Let f_t be a sequence of measure preserving homeomorphisms, and R a measurable set with incoming sets I_t and exit sets E_t. Divide each exit set E_t into a disjoint union of sets with nonzero measure. Then there is a nonzero measure of orbits that enter R at some time $j \leq t$ and leave through each portion of E_t.*

Proof. Let B_t and F_t denote the sets of nonzero measure that form the partition of E_t. Suppose that $\mu(B_t) \geqslant \delta > 0$, but that almost none of the orbits that entered R from the outside leave through B_t. Then the set F_t can be partitioned into disjoint sets $T_k^t = S_k^t \cap F_t$ that entered R at time k. Thus

$$\sum_{k=-\infty}^{t} \mu(T_k^t) \leq \mu(F_t) \ .$$

Since f_t is area-preserving, $\mu(S_k^{t+1}) = \mu(S_k^t \setminus E_t) = \mu(S_k^t) - \mu(T_k^t)$, and we have

$$\begin{aligned} \sum_{k=-\infty}^{t+1} \mu(S_k^{t+1}) &= \sum_{k=-\infty}^{t} \mu(S_k^{t+1}) + \mu(S_{t+1}^{t+1}) \\ &= \sum_{k=-\infty}^{t} \left[\mu(S_k^t) - \mu(T_k^t)\right] + \mu(I_t) \\ &\geq \sum_{k=-\infty}^{t} \mu(S_k^t) - \mu(F_t) + \mu(I_t) = \sum_{k=-\infty}^{t} \mu(S_k^t) + \mu(B_t) \ . \end{aligned}$$

Thus the sequence $\sum_{k=-\infty}^{t} \mu(S_k^t)$ is strictly increasing and therefore unbounded, which violates (18). Thus we contradict the hypothesis that almost nothing leaves through B_t. □

It is not true in this case that almost all entering orbits must leave. For example, for the sequence of maps $\{\dots, f, g, g, g, \dots\}$, f could map a point from outside R onto an invariant set of g that is completely contained in R.

Maps of the Cylinder

We now specialize to the case of area-preserving maps of the cylinder. The net flux, $\mathcal{F}$, is the net area that crosses a homotopically nontrivial circle upon

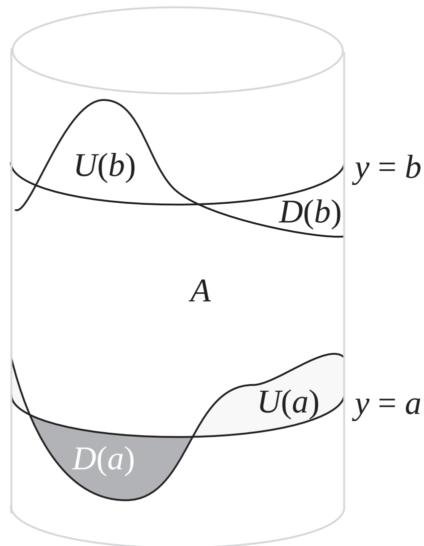

Figure 5: Upward and downward moving sets for a map on the cylinder.

each iteration of map. For any noncontractible loop C, the net flux is defined to be the algebraic area contained between C and its image $f(C)$:

$$\mathcal{F} = \int_C y' dx' - y dx \; .$$

It is easy to see that area-preservation implies that $\mathcal{F}$ is independent of C.

A simple, noncontractible loop C divides the cylinder into two pieces, a top T and bottom B. Let $U \subset T$ denote the "upward moving" set: those points in the top whose preimage is in the bottom

$$U = \{z \in T : f^{-1}(z) \in B\} \; .$$

Similarly, let $D \subset B$ denote the "downward moving" set: those points in the bottom whose preimage is in the top:

$$D = \{z \in B : f^{-1}(z) \in T\} \; .$$

The difference in area $\mu(U) - \mu(D) = \mathcal{F}$ is the net flux.

A simple corollary of Lem. 4 implies that maps with nonzero net flux have orbits that drift arbitrarily far.

Corollary 5. *Suppose that f_t is a sequence of area and end-preserving homeomorphisms of the cylinder, and that the net flux $\mathcal{F}_t \geq \delta > 0$. Let A denote the annulus bounded by the circles $\{y = a\}$ and $\{y = b\}$ where $a < b$. Then there is a set of positive measure of orbits that cross the annulus.*

Proof. Let $U_t(a)$ and $D_t(a)$ be the upward and downward moving sets associated with a curve $\{y = a\}$ for f_t, and similarly for b, see Fig. 5. The annulus A has incoming sets $I_t = U_t(a) \cup D_t(b)$, and exit sets $E_t = f_t^{-1}(D_t(a)) \cup f_t^{-1}(U_t(b))$. Since $\mu(U_t(b)) > 0$, this decomposition satisfies the hypothesis of Lem. 4, and so we know that a nonzero measure of orbits that start below a eventually cross b upon landing in $f_t^{-1}(U_t(b))$. $\square$

Standard Map with Net Flux

Consider a generalized standard map (4) on the cylinder $(0, 2\pi) \times \mathbb{R}$. The net flux is given by $\mathcal{F} = V(2\pi) - V(0)$. In general, the force DV can be separated into its mean and oscillatory parts, giving a constant flux and periodic force, respectively. Thus the general situation is modeled by the standard map:

$$\begin{aligned} x' &= x + y' , \\ y' &= y - k \sin(x) + \frac{1}{2\pi}\mathcal{F} . \end{aligned}$$

When $\mathcal{F} = 0$, and $k < k_{cr} \approx 0.971635406$ this map has rotational invariant circles, and hence all orbits are bounded. The phase space for $k = 0.5$ is shown in Fig. 6. By Cor. 5, whenever $\mathcal{F} \neq 0$ there is a nonzero measure of orbits that cross from $y = 0$ to $y = 2\pi$. By periodicity, these orbits are unbounded. We can visualize these orbits most easily by noting that the standard map can be put on the torus since for any integer m, $f(x, y + 2\pi m) = f(x, y) + 2\pi(m, m)$. In Fig. 7, we show that even when $\mathcal{F}$ is quite small much of the phase space contains unbounded orbits.

Periodic orbits of the zero flux standard map are successively destroyed by saddle-center bifurcations as $\mathcal{F}$ increases. For example, there are fixed points at $y = 0$ and the two branches of

$$x = \arcsin\left(\frac{\mathcal{F}}{2\pi k}\right) ,$$

which collide in a saddle-center bifurcation at $\mathcal{F} = 2\pi k$.

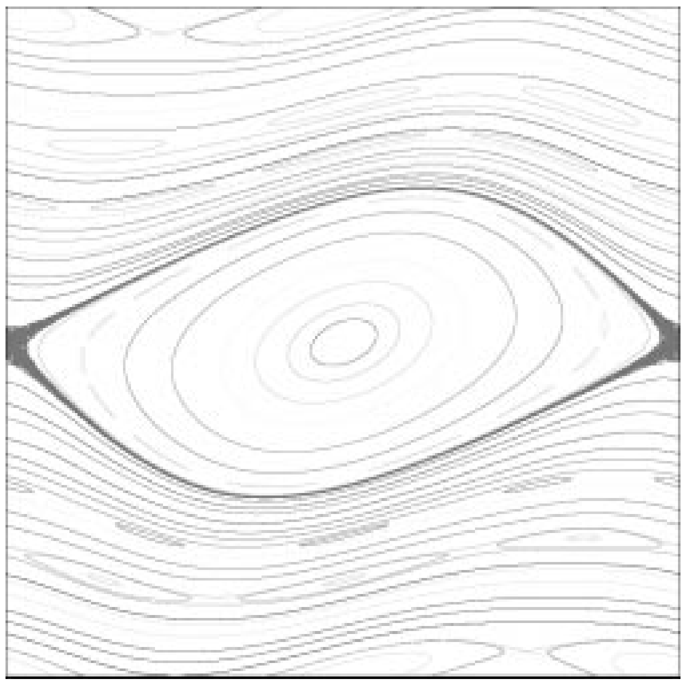

Figure 6: Phase space of the standard map for $k = 0.5$. There are many rotational invariant circles

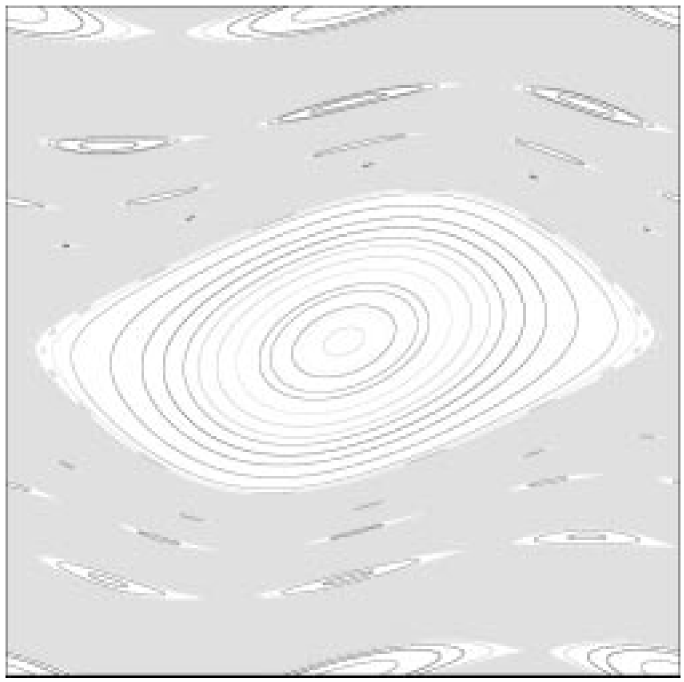

Figure 7: Phase space of the standard map with $k = 0.5$, and net flux $\mathcal{F} = 4\pi^2/1000$. There are no rotational invariant circles, and the majority of phase space is covered by a single orbit (grey region).

6 Persistence of Drift

In this section we study the dynamics of the 4D maps given by (12) for small ε. Here we find it convenient to use the coordinates $z_{t-1} = (x_{t-1}, x_t, \xi_{t-1}, \xi_t)^T$ so that the map becomes

$$z_t = \varphi_\varepsilon(z_{t-1}) = \begin{pmatrix} x_t \\ -x_{t-1} + 2x_t \quad + \quad \frac{1}{\varepsilon} DV(x_t)[1 + C(\xi_t)] \\ \xi_t \\ -\xi_{t-1} + 2\xi_t \quad + \quad DW(\xi_t) + V(x_t) DC(\xi_t) \end{pmatrix} \tag{19}$$

When $\varepsilon = 0$, then any sequence $x_t = c_t \in \mathrm{crit}(V)$ can be used to form an orbit of φ_0. We first show that if the acceleration is bounded, then for $\varepsilon > 0$, there are orbits whose x-coordinates stay close to the critical points forever.

Lemma 6. *Suppose that φ_ε, given by (19), is a C^2 map of $\mathbb{T}^4$, such that $1 + C(\xi) \geqslant \tau > 0$. Then, for any sequence $\{c_0, c_1, \dots\}$ with $c_t \in crit(V) \cap \mathcal{A}$, any initial condition (ξ_0, ξ_1), and any $\delta > 0$, there exists an orbit $z_t = (x_t, x_{t+1}, \xi_t, \xi_{t+1})$, $t \geqslant 0$ of φ_ε such that*

$$|x_t - c_t| \leqslant \delta \text{ for all } t \geqslant 0 \,,$$

provided

$$0 \leqslant \varepsilon < \varepsilon_0 = \sigma\tau/(4\delta + a) \,, \tag{20}$$

where $\sigma(\delta, b) \equiv \inf_{t \geqslant 0} |DV(c_t \pm \delta)|$.

Recall that a is the maximal acceleration in $\mathcal{A}$, (7), and b is the minimal curvature in $\mathrm{crit}(V)$, (5). Note that since V is C^2 we can bound σ by

$$\delta \sup_x |D^2 V(x)| \geqslant \sigma \geqslant \delta \inf_x |D^2 V(x)| \;\;, \;\; x \in \cup_{t \geqslant 0} [c_t - \delta, c_t + \delta] \,.$$

The lower bound is nonzero for small enough δ since the critical points are nondegenerate. Thus $\varepsilon_0 \to 0$ as $\delta \to 0$.

Proof. For each critical point c_t in the sequence, choose c_t^+ to be either $c_t \pm \delta$ in order that $DV(c_t^+) > 0$. Similarly choose c_t^- so that $DV(c_t^-) < 0$. Since the critical points are uniformly nondegenerate, there is a $\sigma(\delta) > 0$ such that $|DV(c_t^\pm)| \geqslant \sigma$. Note that it is possible that $c_t^+ < c_t^-$. However, we will still use $[c_t^-, c_t^+]$ to denote the closed interval between these points.

We start by defining four dimensional "windows" $W_t = [c_t^-, c_t^+] \times [c_{t+1}^-, c_{t+1}^+] \times \mathbb{R}^2$. The "top" of the window W_t is the set $[c_t^-, c_t^+] \times c_{t+1}^+ \times \mathbb{R}^2$, and the region "above" the window is the set $U_t = [c_t^-, c_t^+] \times (c_{t+1}^*, \infty) \times \mathbb{R}^2$ where c_{t+1}^* is the maximum of c_{t+1}^- and c_{t+1}^+.

It suffices to show that there exists an orbit z_t of φ with $z_t \in W_t$ for each $t > 0$ and with $z_0 = (x_0, x_1, \xi_0, \xi_1)$. Note that φ maps W_t into the set $[c_{t+1}^-, c_{t+1}^+] \times \mathbb{R}^3$, see Fig. 8.

Choose a point $(x_t, c_{t+1}^+, \xi_t, \xi_{t+1})$ in the top of W_t. We show that φ maps this point into the region U_{t+1} above the next window. Let $\varphi(x_t, c_{t+1}^+, \xi_t, \xi_{t+1}) = (c_{t+1}^+, x_{t+2}, \xi_{t+1}, \xi_{t+2})$. Thus we need only show that the second coordinate $x_{t+2} > c_{t+2}^*$. By (19),

$$x_{t+2} = 2c_{t+1}^+ - x_t + \frac{1}{\varepsilon} DV(c_{t+1}^+)[1 + C(\xi_{t+1})] \ ,$$

thus we must show that

$$(1/\varepsilon) DV(c_{t+1}^+)[1 + C(\xi_{t+1})] > c_{t+2}^* - 2c_{t+1}^+ + x_t \ .$$

Since x_t, c_{t+1}^+ and c_{t+2}^* and are all within δ of the respective values c_t, c_{t+1} and c_{t+2}, it is sufficient to show

$$\frac{1}{\varepsilon} DV(c_{t+1}^+)[1 + C(\xi_{t+1})] > 4\delta + c_{t+2} - 2c_{t+1} + c_t \ .$$

Since the final three terms are bounded by a and since $1 + C(\xi) \geqslant \tau$ by assumption, we need to require that $\frac{1}{\varepsilon} DV(c_{t+1}^+)\tau > (4\delta + a)$. Thus the final condition (on ε) is the condition $\sigma\tau/\varepsilon > 4\delta + a$ or equivalently (20).

A similar argument shows that the bottom of the window W_t maps into the region B_{t+1} below the window W_{t+1}.

Let S be a "slice" of the window W_0 defined by $S = \mathbb{R}^2 \times (\xi_0, \xi_1) \cap W_0$. It is sufficient to show that the orbit of some point z_0 in S stays inside the sequence of windows. Let T be the set of points in S which exit the window

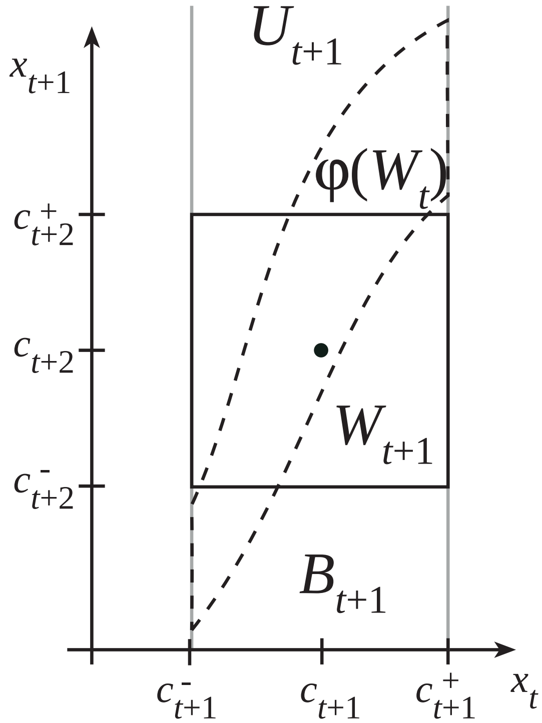

Figure 8: The image of a four dimensional window, W_t under φ crosses the window W_{t+1}. We suppress both of the ξ coordinates.

sequence by landing in some set U_t above the window W_t for some $t \geqslant 1$. Similarly let B denote the set of points which exit the window sequence by falling below some window. Since the set S intersects the top and bottom of the window W_0, these sets T and B are non-empty.

We note that the sets T and B are also open sets. This is because the top and bottom of any window W_t maps above or below the next window. Therefore, since the slice S is connected, it is not the disjoint union of T and B. Therefore there exist some points in S that remain in the sequence of windows W_t forever. $\square$

By Thm. 2 there is a nonzero measure of points (ξ_0, ξ_1) such that there exists a sequence $\{c_t\} \in \operatorname{crit}(V)$ for which an "orbit" $Z_t = (c_t, c_{t+1}, \xi_t, \xi_{t+1})$ at the anti-integrable limit has momenta that drift arbitrarily far. We show next that for ε sufficiently small, the orbit $\{Z_t\}$ is "shadowed" by an orbit $\{z_t\}$ of the 4-D map φ for some arbitrarily long time T if ε is small enough.

Theorem 7. *Suppose that φ_ε satisfies the hypotheses of Lem. 6. Let $Z_t = (c_t, c_{t+1}, \xi_t, \xi_{t+1})$ be an orbit of φ_0 with $c_t \in crit(V)$. Then for any $T \geqslant 0$ and $\gamma > 0$, there is a $\delta > 0$ such that for all $\varepsilon < \varepsilon_0(\delta)$ in (20) there is an orbit $z_t = (x_t, x_{t+1}, \zeta_t, \zeta_{t+1})$ of φ_ϵ such that $|\zeta_t - \xi_t| < \gamma$ for all $0 \leqslant t \leqslant T$.*

Proof. Chose δ and ε obeying (20). By Lem. 6 there is an orbit $z_t = (x_t, x_{t+1}, \zeta_t, \zeta_{t+1})$ of φ_ε with $\zeta_0 = \xi_0$, $\zeta_1 = \xi_1$ and $|x_t - c_t| \leq \delta$ for all t. We have

$$\xi_{t+1} - 2\xi_t + \xi_{t-1} = V(c_t)DC(\xi_t) + DW(\xi_t) \ ,$$

and

$$\zeta_{t+1} - 2\zeta_t + \zeta_{t-1} = V(x_t)DC(\zeta_t) + DW(\zeta_t) \ .$$

Let $\lambda_t = \xi_t - \zeta_t$. Then $\lambda_{t+1} - 2\lambda_t + \lambda_{t-1} = V(c_t)DC(\xi_t) - V(x_t)DC(\zeta_t) + DW(\xi_t) - DW(\zeta_t)$, and $\lambda_0 = \lambda_1 = 0$. Since V is C^2, and DC is bounded, we have

$$|V(c_t)DC(\xi_t) - V(x_t)DC(\zeta_t)| \leqslant \frac{1}{2}M\delta^2 \ ,$$

where $M = \max_x |D^2V(x)| \ \max_\xi |DC(\xi)|$ such that $x \in \cup_{t\geqslant 0}[c_t - \delta, c_t + \delta]$ and $\xi \in \mathbb{T}$. Since W is C^2 as well, λ_t obeys the inequality

$$|\lambda_{t+1} - (2 + DW(\tilde{x}_t))\lambda_t + \lambda_{t-1}| \leqslant \frac{1}{2}M\delta^2 \ ,$$

where $\tilde{x}_t$ is some point in $[c_t, x_t]$. It is straightforward to see that the solutions of this second difference inequality are bounded by

$$|\lambda_t| \leqslant \frac{1}{2}M\delta^2 \sum_{\tau=1}^{t-1} \left(|\lambda_t^1\lambda_\tau^2| + |\lambda_\tau^1\lambda_t^2|\right) \ ,$$

where λ_t^1, and λ_t^2 are linearly independent solutions of the homogeneous equation normalized so that their Wronskian $\lambda_1^1\lambda_0^2 - \lambda_0^1\lambda_1^2 = 1$. For example, we may choose solutions so that $\lambda_1^1 = \lambda_0^2 = 1$ and $\lambda_0^1 = \lambda_1^2 = 0$. These can be easily bounded by $|\lambda_t^i| \leqslant r^t$, where $r > 1$ is the positive root of $r^2 - wr - 1 = 0$, and $w = \max_x(2 + |DW(x)|)$. Thus we obtain the bound

$$|\lambda_t| \leqslant \frac{1}{2} M\delta^2 r^{2t} .$$

This result can be made stronger when $DW = 0$, because in this case λ_t can only grow as t^2—see below. Thus for any T and any γ we can choose δ small enough so that $|\lambda_t| \leqslant \gamma$ for $t \leqslant T$. □

Remark: The orbit ξ_t in the anti-integrable limit is not necessarily hyperbolic and thus one does not expect to be able to shadow such an orbit for more than a finite interval.

Since the orbit of the anti-integrable system can be chosen to have drifting momenta, this result implies that for small enough ϵ, there are orbits of the full system whose momenta grow by an arbitrarily large amount. Note that this is true even when the coupling $C(\xi)$, is arbitrarily small.

Standard Example, Continued

Theorem 7 applies to the example (15), and we can compute the bounds explicitly. Since $V(x) = k\cos x$, if we assume $\delta < \frac{\pi}{2}$, we find $\sigma = k\sin\delta$. Since $C(\xi) = h\cos\xi$, we have $\tau = 1 - h$, so we require $h < 1$. We can choose the critical points $c_t \in \{0, \pi\}$, so that the (x, y) orbit does not undergo any rotations. In this case the acceleration of an arbitrary sequence of critical points is at most $a = 2\pi$. Then Lem. 6 applies for

$$\varepsilon \leqslant \varepsilon_0 = \frac{k(1-h)\sin\delta}{4\delta + 2\pi} .$$

The bound in Thm. 7 is obtained by setting $M = kh$ and noting that since $W = 0$, we can use the homogeneous solutions $\lambda_t^1 = t$, and $\lambda_t^2 = 1$, so that

$$|\lambda_t| \leqslant \frac{3}{4} kh\delta^2 t(t-1) ,$$

Thus it is sufficient, for shadowing within γ up to time T, to choose

$$\delta \leqslant \frac{1}{T}\sqrt{\frac{\gamma}{kh}} ,$$

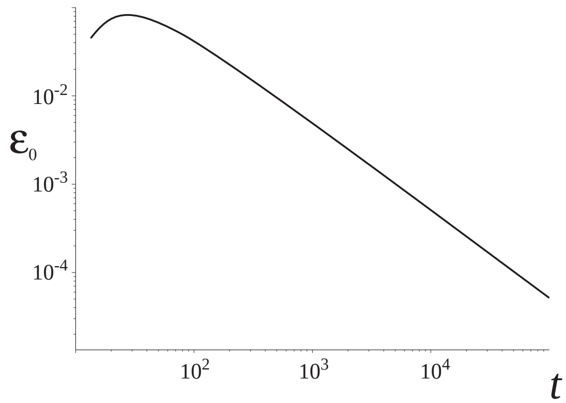

Figure 9: Bound on ε for the standard example with $k = 1$, $h = 0.001$, for a deviation of $\gamma = 1$.

providing $h < 1$ and $\varepsilon \leqslant \varepsilon_0$. We show a sketch of the bound on ε in Fig. 9. We expect that this bound is overly restrictive, and that the drift will actually persist farther.

7 Conclusions

We have shown that a twist map coupled to a map that is near an anti-integrable limit has many orbits whose momenta drift arbitrarily far—even when the coupling is arbitrarily small. Our analysis applies only to the case of "a priori chaotic" systems [DdlLS00] as all the continued orbits from an anti-integrable limit are hyperbolic.

The separation of a system into an essentially chaotic degree of freedom that drives an essentially integrable degree of freedom is a common technique used in physical calculations of Arnold diffusion rates [CLSV84, Viv84, LL92]. In this sense our calculation can be thought of as treating the case of "thick layer" diffusion, rather than the essentially more difficult "thin layer" case, where the chaotic motion is exponentially slow. In the calculations, the coupling—which necessarily affects both degrees of freedom for a symplectic system—is treated by what is called the "stochastic pump model" of Tennyson et al. [TLL79]. In our analysis this is essentially equivalent to the semidirect product at the anti-integrable limit. Our results can be viewed as a step along the road to validating the stochastic pump model.

Of course, we have not provided a solution to the difficult problem: is Arnold diffusion generic in near-integrable systems? Nor have we shown that our drift is actually diffusive. To this end, Moeckel has shown that an ergodic theorem applies in some cases when the full shift is coupled via a semidirect product to an area-preserving map [Moe00].

The approach we have presented might be generalizable to the case when the dynamics in the (x, y) system is not assumed to be conjugate to a full shift, as in the anti-integrable limit, but rather to some subshift. In this case, one is no longer able to drive the drift monotonically, but drift may nevertheless occur. Our approach may also be useful for numerical computations, since the anti-integrable limit is an effective point at which to begin continuation methods [SDM99].

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