Reversors and Symmetries for Polynomial Automorphisms of the Plane

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Abstract

We obtain normal forms for symmetric and for reversible polynomial automorphisms (polynomial maps that have polynomial inverses) of the plane. Our normal forms are based on the generalized Hénon normal form of Friedland and Milnor. We restrict to the case that the symmetries and reversors are also polynomial automorphisms. We show that each such reversor has finite order, and that for nontrivial, real maps, the reversor has order 2 or 4. The normal forms are shown to be unique up to finitely many choices. We investigate some of the dynamical consequences of reversibility, especially for the case that the reversor is not an involution.

1   Introduction

Polynomial maps provide one of the simplest, nontrivial classes of nonlinear dynamical systems. A subset of these, called *polynomial automorphisms*, have polynomial inverses—thus these maps are diffeomorphisms. Since this subset is closed under composition, polynomial automorphisms form a group that we denote by $\mathcal{G}$. These maps can have quite complicated dynamics, as exemplified by the renowned Hénon quadratic map [1, 2], which is in $\mathcal{G}$. The family of generalized Hénon maps [3],

$$ h(x, y) = (y, p(y) - \delta x) \,, \tag{1} $$

for any polynomial p, is also in $\mathcal{G}$ whenever $\delta \neq 0$, since

$$h^{-1}(x, y) = \left(\delta^{-1}(p(x) - y), x\right) \ ,$$

is also polynomial. These maps are area-preserving when $\delta = \pm 1$, and orientation-preserving when $\delta > 0$.

The structure of $\mathcal{G}$ is well understood, thanks to a classic result of Jung [4]. In this paper we will make use of Jung's theorem to investigate polynomial automorphisms that have symmetries or reversing symmetries in $\mathcal{G}$. Our results also extensively use the normal form for polynomial automorphisms as compositions of generalized Hénon maps obtained by Friedland and Milnor [3].

A diffeomorphism g has a symmetry if it is conjugate to itself; that is, there exists a diffeomorphism S such that

$$g = S^{-1} \, g \, S \ . \tag{2}$$

Similarly, g has a reversing symmetry, or is "reversible", if it is conjugate to its inverse [5, 6, 7, 8]; that is, there exists a diffeomorphism R such that

$$g^{-1} = R^{-1} \, g \, R \ . \tag{3}$$

For example, generalized Hénon maps are reversible when $\delta = 1$, and they have a nontrivial symmetry when there is an $\omega \neq 1$ such that $p(\omega y) = \omega p(y)$. Reversible maps occur often in applications. For example, reversibility often arises for Hamiltonian systems because their phase spaces consist of coordinates q and momenta p, and a transformation that reverses the momenta, $R(q, p) = (q, -p)$, often corresponds to reversal of time. Our goal in this paper is to classify the polynomial automorphisms that satisfy (2) or (3) with $S, R \in \mathcal{G}$.

The basic properties of symmetries and reversors are discussed in [9] and reviewed in [8]. The set of symmetries of g is never empty since the identity always satisfies (2). Moreover, since the composition of any two symmetries is also a symmetry, they form a group

$$\mathrm{Sym}(g) = \{S \in \mathcal{G} : S^{-1} \, g \, S = g\} \ . \tag{4}$$

In group theory terminology, $\mathrm{Sym}(g)$ is the *centralizer* of g in $\mathcal{G}$. Note that the family $\{g^j : j \in \mathbb{Z}\}$ is a subgroup of $\mathrm{Sym}(g)$. We will say that S is a nontrivial symmetry of g in the case that $S \neq g^j$. If all of the symmetries of g are trivial then $\mathrm{Sym}(g)$ is isomorphic to $\mathbb{Z}$ (or to $\mathbb{Z}_n$ if g has finite order).

Similarly, whenever R is a reversor for g, then so are each of the members of the family

$$\{R_{j,k} = g^j R^{2k+1} : j, k \in \mathbb{Z}\} \ , \tag{5}$$

Thus if R is a reversor, then its inverse is one as well. On the other hand, the composition of any two reversors is a symmetry of g (and is not a reversor unless g is an involution). Thus for example, if R is a reversor then R^2 is a symmetry. Moreover, the composition of a symmetry and of a reversing symmetry of g yields a reversing symmetry. It follows that the

set of all symmetries and reversing symmetries of g is a group, usually referred as the group of *reversing symmetries* of g

$$\mathrm{Rev}(g) = \{f \in \mathcal{G} : f^{-1}\, g\, f = g^{\pm 1}\}. \tag{6}$$

The group $\mathrm{Sym}(g)$ is a normal subgroup of $\mathrm{Rev}(g)$ and $\mathrm{Rev}(g)/\mathrm{Sym}(g)$ is isomorphic to $\mathbb{Z}_2$, the permutation group with two elements. The properties of the group of reversing symmetries have been investigated in several recent papers [10, 11, 12, 13] as well as the papers in the collection [14].

Reversors arising in physical examples often are involutions, $R^2 = id$, so that R generates a group $\langle R \rangle = \{id, R\} \simeq \mathbb{Z}_2$. If g possesses an involutory reversing symmetry R, then $\mathrm{Rev}(g) = \mathrm{Sym}(g) \rtimes \langle R \rangle$ [10];[1] therefore, in this case it suffices to determine the structure of $\mathrm{Sym}(g)$ to obtain a full description of $\mathrm{Rev}(g)$. However, reversors need not be involutions; examples of maps with noninvolutory reversors (called weakly reversible by Sevryuk [6]) were given by Lamb [9]. As we will see in §4, any reversor in $\mathrm{Rev}(g)$ has finite order, $R^k = id$. Note that if k were odd, then g would itself be an involution and thus is dynamically trivial; so if we restrict to nontrivial maps in $\mathcal{G}$, then their reversors have even order. Moreover, for real maps and reversors, we will show that the order must be two or four. After we submitted the current paper, we learned that Roberts and Baake have also announced this result for real maps in $\mathcal{G}$ [13].

We will also demonstrate that if g has a nontrivial symmetry, then there is a subgroup of $\mathrm{Sym}(g)$ generated by an finite order map and a "root" of g. In particular, we will show

Theorem 1. *Suppose g is a polynomial automorphism of the plane that possesses nontrivial symmetries. Then g is conjugate to a map of the form*

$$s(H)^q \,,$$

where H is a composition of generalized Hénon maps (1), s is a diagonal linear map with finite order, $s^n = id$, and either $s \neq id$ or $q \neq 1$. The normal form has commuting symmetries s and H, and $Sym(G) \supset \langle H \rangle \times \langle s \rangle \simeq \mathbb{Z} \times \mathbb{Z}_n$.

This theorem is stated in a more detailed way in §3 as Thm. 7 and Cor. 9. Note that for the real case, s is at most order two (this result was also announced in [13]).

In a previous paper we obtained normal forms for the automorphisms that are reversible by an involution in $\mathcal{G}$ [15]. Just as in [3], these normal forms are constructed from compositions of generalized Hénon maps; however, in this case, the reversors are introduced by including two involutions in this composition. We showed there that the involutions can be normalized to be either "elementary" involutions, or the simple affine permutation

$$t(x,y) = (y,x) \,. \tag{7}$$

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[1]Recall that the semidirect product $\mathcal{G} = \mathcal{N} \rtimes \mathcal{M}$ where $\mathcal{N}$ is a normal subgroup of $\mathcal{G}$ is defined so that if $g = (n,m) \in \mathcal{G}$ then the product is $g_1\, g_2 = (n_1\, m_1\, n_2\, m_1^{-1}, m_1\, m_2)$. This contrasts with the direct product $\mathcal{N} \times \mathcal{M}$ where $g_1\, g_2 = (n_1\, n_2, m_1\, m_2)$.

The second major result of the current paper is that in the general case, reversible automorphisms also have at least two basic reversors that are also either elementary or affine. In particular we will prove

Theorem 2. *Suppose g is a nontrivial reversible automorphism. Then g possesses a reversor of order $2n$ in $\mathcal{G}$ and is conjugate to one of the following classes:*

$$\mathbf{R_{AA}} \qquad \tau_\omega^{-1} \, H^{-1} \, \tau_\omega \, H,$$

$$\mathbf{R_{AE}} \qquad \tau_\omega^{-1} \, H^{-1} \, e_2 \, H,$$

$$\mathbf{R_{EE}} \qquad e_1^{-1} \, t \, H^{-1} \, e_2 \, H \, t,$$

where

- *the map H is a composition of generalized Hénon transformations (1),*

- *τ_ω is the affine reversor, $\tau_\omega(x, y) = (\omega y, x)$, such that ω is a primitive n^{th} root of unity, and*

- *the maps e_1, e_2 are elementary reversors,*

$$e_i(x, y) = (p_i(y) - \delta_i \, x, \epsilon_i \, y)$$

  where $p_i(\epsilon_i \, y) = \delta_i \, p_i(y)$ and ϵ_i^2, δ_i^2 are primitive n^{th} roots of unity.

As we will see in §4, the reversor is an involution ($n = 1$) unless the polynomials in the Hénon maps and the elementary transformations satisfy a common scaling condition. In this case we will see that ω, ϵ_i^2 and δ_i^2 lie in subsets of the nth roots of unity that we will explicitly construct.

We also show in §4 that the maps in Thm. 2 can be normalized, and that once this is done the normal forms are unique up to finitely many choices. These details are contained in the more complete statement, Thm. 12.

We finish the paper with a discussion of some examples and their dynamics.

2 Background

We start by giving some definitions and basic results concerning the algebraic structure of the group $\mathcal{G}$, presenting our notation, and reviewing the work of Friedland and Milnor [3].

Jung's Theorem

The group $\mathcal{G}$ is the group of *polynomial automorphisms* of the complex plane, the set of bijective maps

$$g : (x, y) \to (X(x, y), Y(x, y)), \quad X, Y \in \mathbb{C}[x, y] \,,$$

having a polynomial inverse. Here $\mathbb{C}[x, y]$ is the ring of polynomials in the variables x and y, with coefficients in $\mathbb{C}$. In general, we consider this complex case, but in some instances we will restrict to the case of real maps. The *degree* of g is defined as the largest of the degrees of X and Y.

The subgroup $\mathcal{E} \subset \mathcal{G}$ of *elementary* (or triangular) maps consists of maps of the form

$$e : (x, y) \to (\alpha x + p(y), \beta y + \eta) , \tag{8}$$

where $\alpha\beta \neq 0$ and $p(y)$ is any polynomial. The subgroup of affine automorphisms is denoted by $\mathcal{A}$. The affine-elementary maps will be denoted by $\mathcal{S} = \mathcal{A} \cap \mathcal{E}$.

We let $\hat{\mathcal{S}}$ denote the group of diagonal affine automorphisms,

$$\hat{s} : (x, y) \to (\alpha x + \xi, \beta y + \eta) . \tag{9}$$

The group $\hat{\mathcal{S}}$ is the largest subgroup of $\mathcal{S}$ normalized by the permutation t (7), i.e., such that $t\, \hat{\mathcal{S}}\, t = \hat{\mathcal{S}}$. On the other hand, the centralizer of t in $\mathcal{S}$ is the subgroup of maps that commute with t

$$C_{\mathcal{S}}(t) = \{s \in \mathcal{S} : s\, t\, s^{-1} = t\} .$$

These are the diagonal automorphisms (9) with $\alpha = \beta$ and $\xi = \eta$. Conjugacy by t will be denoted by ϕ,

$$\phi(g) = t\, g\, t. \tag{10}$$

Thus if $s \in C_{\mathcal{S}}(t)$, then $\phi(s) = s$.

According to Jung's Theorem [4] every polynomial automorphism $g \notin \mathcal{S}$, can be written as

$$g = g_m\, g_{m-1} \cdots g_2\, g_1, \qquad g_i \in (\mathcal{E} \cup \mathcal{A}) \setminus \mathcal{S}, \ i = 1, \ldots m , \tag{11}$$

with consecutive terms belonging to different subgroups $\mathcal{A}$ or $\mathcal{E}$. An expression of the form (11) is called a *reduced word* of length m. An important property of a map written in this form is that its degree is the product of the degrees of the terms in the composition [3, Thm. 2.1]. A consequence of this fact is that the identity cannot be expressed as a reduced word [3, Cor. 2.1]. This means that $\mathcal{G}$ is the free product of $\mathcal{E}$ and $\mathcal{A}$ amalgamated along $\mathcal{S}$. The structure of $\mathcal{G}$ as an amalgamated free product determines the way in which reduced words that correspond to the same polynomial automorphism, are related.

Theorem 3. *(cf. [3, Cor. 2.3], or [16, Thm. 4.4]) Two reduced words $g_m \cdots g_1$ and $\tilde{g}_n \cdots \tilde{g}_1$ represent the same polynomial automorphism g if and only if $n = m$ and there exist maps $s_i \in \mathcal{S}$, $i = 0, \ldots, m$ such that $s_0 = s_m = id$ and $\tilde{g}_i = s_i\, g_i\, s_{i-1}^{-1}$.*

From this theorem it follows that the length of a reduced word (11) as well as the degrees of its terms are uniquely determined by g. The sequence of degrees $(l_1, \ldots, l_n)$ corresponding to the maps $(g_1, \ldots, g_m)$, after eliminating the 1's coming from affine terms, is referred to as the *polydegree* of g.

A map is said to be *cyclically-reduced* in the trivial case that it belongs to $\mathcal{A} \cup \mathcal{E}$ or when it can be written as a reduced word (11) with $m \geq 2$ and g_m, g_1 not in the same subgroup $\mathcal{E}$ or $\mathcal{A}$.

5

Conjugacies of Polynomial Autormorphims

Two maps $g, \tilde{g} \in \mathcal{G}$ are conjugate in $\mathcal{G}$ if there exists $f \in \mathcal{G}$ such that $g = f\tilde{g}f^{-1}$. If f belongs to some subgroup $\mathcal{F}$ of $\mathcal{G}$ we say that g and $\tilde{g}$ are $\mathcal{F}$-conjugate. It can be easily seen that every $g \in \mathcal{G}$ is conjugate to a cyclically-reduced map. Moreover, an explicit calculation shows that that every affine map a can be written as $a = s\,t\,\tilde{s}$, where t is given by (7) and $s, \tilde{s}$ affine-elementary maps. From these facts it follows that every polynomial automorphism is either *trivial* (i.e., conjugate to an elementary or an affine map) or is conjugate to a reduced word of the form,

$$g = t\,e_m \cdots t\,e_2\,t\,e_1, \quad e_i \in \mathcal{E} \setminus \mathcal{S}, \quad i = 1, \cdots m, \quad m \geq 1. \tag{12}$$

Moreover this representative of the conjugacy class is unique up to modifications of the maps e_i by diagonal affine automorphisms and cyclic reordering. More precisely we have the following theorem (following [16, Thm. 4.6]).

Theorem 4. *Two nontrivial, cyclically-reduced words $g = g_m \ldots g_1$ and $\tilde{g} = \tilde{g}_n \ldots \tilde{g}_1$ are $\mathcal{G}$-conjugate if and only if $m = n$ and there exist automorphisms $s_i \in \mathcal{S}$, $i = 0, \ldots, m$ with $s_m \equiv s_0$, and a cyclic permutation,*

$$(\hat{g}_m, \ldots, \hat{g}_1) = (\tilde{g}_k, \ldots, \tilde{g}_1, \tilde{g}_m, \ldots, \tilde{g}_{k+1})$$

such that $\hat{g}_i = s_i\,g_i\,s_{i-1}^{-1}$. In that case,

$$s_0\,g\,s_0^{-1} = \hat{g}_m \ldots \hat{g}_1.$$

In particular, if $g = t\,e_m \cdots t\,e_1$ and $\tilde{g} = t\,\tilde{e}_m \cdots t\,\tilde{e}_1$ are conjugate, there exist diagonal automorphisms $s_i \in \hat{\mathcal{S}}$, $s_m \equiv s_0$, and a cyclic reordering,

$$(\hat{e}_m, \ldots, \hat{e}_1) = (\tilde{e}_k, \ldots, \tilde{e}_1, \tilde{e}_m \ldots, \tilde{e}_{k+1}),$$

such that $t\,\hat{e}_i = s_i\,t\,e_i\,s_{i-1}^{-1}$ and

$$s_0\,g\,s_0^{-1} = t\,\hat{e}_m \cdots t\,\hat{e}_1.$$

Proof. Let $g = g_m \ldots g_1$ and $\tilde{g} = \tilde{g}_n \ldots \tilde{g}_1$ be two nontrivial, cyclically-reduced, conjugate words. By assumption, there is a reduced word $f = f_k \cdots f_1 \in \mathcal{G}$, such that $g = f\tilde{g}\,f^{-1}$. Then,

$$g_m \cdots g_1 = f_k \cdots f_1\,\tilde{g}_n \cdots \tilde{g}_1\,f_1^{-1} \cdots f_k^{-1}. \tag{13}$$

However, the word on the right hand side of (13) is not reduced. Since $\tilde{g}$ is cyclically-reduced, we can suppose, with no loss of generality, that f_1 and $\tilde{g}_1$ belong to the same subgroup $\mathcal{A}$ or $\mathcal{E}$, so that f_1 and $\tilde{g}_n$ lie in different subgroups. Taking into account Thm. 3 and that (13) represents a cyclically-reduced map, we can reduce to obtain

$$\tilde{g}_n \cdots \tilde{g}_1\,f_1^{-1} \cdots f_k^{-1} = \begin{cases} \tilde{g}_n \cdots \tilde{g}_{k+1}\,\tilde{s}_k & \text{if } n \geq k \\ \tilde{s}_n\,f_{n+1}^{-1} \cdots f_k^{-1} & \text{if } n < k, \end{cases} \tag{14}$$

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where $\tilde{s}_n, \tilde{s}_k \in \mathcal{S}$. Moreover there exist $\tilde{s}_i \in \mathcal{S}$, $\tilde{s}_0 = id$, such that $\tilde{g}_i \, \tilde{s}_{i-1} \, f_i^{-1} = \tilde{s}_i$, for $i = 1, \ldots, \min(n, k)$.

For the case $n \geq k$,

$$
\begin{aligned}
g_m \, \cdots \, g_1 &= f_k \, \cdots \, f_1 \, \tilde{g}_n \, \cdots \, \tilde{g}_{k+1} \, \tilde{s}_k \\
&= (\tilde{s}_k^{-1} \, \tilde{g}_k) \, \tilde{g}_{k-1} \, \cdots \, \tilde{g}_1 \, \tilde{g}_n \, \ldots \, \tilde{g}_{k+2} \, (\tilde{g}_{k+1} \, \tilde{s}_k),
\end{aligned}
$$

and applying Thm. 3 we have the result. The case $n < k$ follows analogously.

To prove the second statement of this theorem it is enough to recall that given $s \in \mathcal{S}$, $t \, s \, t$ stays in $\mathcal{S}$ if and only if s is diagonal. $\qquad\square$

This implies that the length of a cyclically-reduced word is an invariant of the conjugacy class. Since a nontrivial, cyclically-reduced word has the same number of elementary and affine terms, we refer to this number as the *semilength* of the word. Thm. 4 also implies that two cyclically-reduced maps that are conjugate have the same polydegree up to cyclic permutations. We will call this sequence the polydegree of the conjugacy class.

Generalized Hénon transformations

A *generalized Hénon transformation* is any map of the form (1) where $\delta \neq 0$ and $p(y)$ is a polynomial of degree $l \geq 2$. Notice that a generalized Hénon transformation can be written as the composition

$$
h = t \, e \,, \quad e(x, y) = (p(y) - \delta x, y) \,.
$$

If $p(y)$ has leading coefficient equal to 1 (± 1 in the case of real automorphisms) and center of mass at 0,

$$
p(y) = y^l + O(y^{l-2}) \,, \tag{15}
$$

we say that the polynomial is *normal*, and consequently that the Hénon transformation is *normalized*. In [3], Friedland and Milnor obtained normal forms for conjugacy classes of elements in $\mathcal{G}$, in terms of generalized Hénon transformations.

Theorem 5. *[3, Thm. 2.6] Every nontrivial $g \in \mathcal{G}$ is conjugate to a composition of generalized Hénon transformations, $h_m \, \cdots \, h_1$. Additionally it can be required that each of the terms h_i be normalized and in that case the resulting normal form is unique, up to finitely many choices.*

To prove this result it is enough to take Thm. 4 into account and check that, given a map $g = t \, e_m \, \cdots \, t \, e_i$, it is possible to choose diagonal affine automorphisms s_i, $i = 1, \ldots, m$, in such a way that s_m coincides with s_0, and for every i, $s_i \, t \, e_i \, s_{i-1}^{-1}$ is a normalized Hénon transformation. In the next section we will use the following generalization of Thm. 5.

Lemma 6. *Given a cyclically-reduced map of the form (12), there exist diagonal affine automorphisms $s_m, s_0 \in C_{\mathcal{S}}(t)$, such that*

$$
s_m \, g \, s_0^{-1} = h_m \, \cdots \, h_1,
$$

where every term h_i is a normal Hénon transformation. The Hénon maps are unique up to finitely many choices.

Proof. Consider a cyclically reduced map (12), with

$$t\,e_i : (x, y) \rightarrow (\beta_i\, y + \eta_i, \alpha_i\, x + p_i(y)), \qquad i = 1, \ldots, m.$$

We look first for diagonal affine maps s_i, $i = 0, \ldots, m$, with $s_0, s_m \in C_{\mathcal{S}}(t)$ and such that the maps $t\,\hat{e}_i = s_i\, t\, e_i\, s_{i-1}^{-1}$ are Hénon transformations. If we denote

$$s_i(x, y) = (u, v) = (a_i x + b_i, c_i y + d_i),$$

the problem reduces to the set of equations,

$$a_0 = c_0, \qquad b_0 = d_0, \qquad a_m = c_m, \qquad b_m = d_m,$$

and

$$\beta_i a_i = c_{i-1}, \qquad b_i = d_{i-1} - \eta_i a_i, \qquad i = 1, \ldots, m.$$

This system can be easily solved in terms of $2m$ parameters; a particular solution is obtained by choosing $c_0 = \cdots = c_{m-1} = 1$ and $d_0 = \cdots = d_{m-1} = 0$. We can assume now that the terms $t\,e_i$ in (12) are already Hénon maps, but not necessarily in normal form. In that case,

$$t\,e_i : (x_{i-1}, x_i) \rightarrow (x_i, x_{i+1}), \quad x_{i+1} = p_i(x_i) - \delta_i x_{i-1}.$$

Setting $s_i\,(x_i, x_{i+1}) = (y_i, y_{i+1})$, $y_i = a_i x_i + b_i$ and $t\,\hat{e}_i = s_i\,(t\,e_i)\,s_{i-1}^{-1}$, we have,

$$t\,\hat{e}_i : (y_{i-1}, y_i) \rightarrow (y_i, y_{i+1}), \quad y_{i+1} = \hat{p}_i(y_i) - \hat{\delta}_i y_{i-1},$$

$$\hat{p}_i(y) = a_{i+1} p_i \left(\frac{y - b_i}{a_i} \right) + \text{const.}, \quad \hat{\delta}_i = \frac{a_{i+1}}{a_{i-1}} \delta_i.$$

In order to have leading coefficients equal to 1 we need

$$\kappa_i a_{i+1} = a_i^{l_i}, \quad i = 1, \ldots, m,$$

where κ_i is the leading coefficient of p_i. On the other hand we require $a_{m+1} = a_m$, $a_1 = a_0$, since by assumption $s_m, s_0 \in C_{\mathcal{S}}(t)$. It is easy to see that these conditions yield a_1 up to l^{th}-roots of unity, where $l = l_1 \cdots l_{m-1}(l_m - 1)$. All other a_i are then uniquely determined. Finally, the coefficients b_i, $i = 1, \ldots, m$, can be chosen so that the next to highest order terms are equal to zero and we set $b_0 = b_1$ and $b_{m+1} = b_m$ to ensure that s_0, s_m are in the centralizer of t.

The above arguments also show that the terms $t\,\hat{e}_i$ are unique up to replacing the polynomials $\hat{p}_i(y)$ and parameters $\hat{\delta}_i$ with $\zeta^{l_i \cdots l_0} \hat{p}_i(y / \zeta^{l_{i-1} \cdots l_0})$ and $\zeta^{l_i l_{i-1}} \hat{\delta}_i$ respectively, where ζ is any lth- root of unity and $l_0 = 1$. $\qquad\square$

Roots of Unity

As the proof of Lem. 6 shows, the normal forms for polynomial automorphisms are unique only up to a scaling by certain roots of unity. In subsequent sections, we will see that symmetric and reversible automorphisms are associated with several subgroups of the roots of unity. In anticipation of these results, we provide some notation for these subgroups.

Let $\mathcal{U}$ be the group of all roots of unity in $\mathbb{C}$ (the points with rational angles on the unit circle) and $\mathcal{U}_n$ be the group of nth-roots of unity:

$$\mathcal{U}_n \equiv \{z \in \mathbb{C} : z^n = 1\} . \tag{16}$$

Given a sequence of normal (15) (nonlinear) polynomials, $p_1(y), \ldots, p_m(y)$, and some root of unity ζ, define the set $\mathcal{R}(\zeta) = \mathcal{R}(\zeta; p_1(y), \ldots, p_m(y))$, by

$$\mathcal{R}(\zeta) \equiv \{\omega \in \mathbb{C} : \omega\, p_{2i+1}(\omega\, y) = \zeta\, p_{2i+1}(y),\ \omega\, p_{2i}(\omega\, y) = p_{2i}(\zeta\, y),\ 1 \leq 2i, 2i+1 \leq m\} . \tag{17}$$

It can be observed that if $\omega \in \mathcal{R}(\zeta)$ then $\omega^{-1} \in \mathcal{R}(\zeta^{-1})$, while if $\omega_1 \in \mathcal{R}(\zeta_1)$ and $\omega_2 \in \mathcal{R}(\zeta_2)$ it follows that $\omega_1\, \omega_2 \in \mathcal{R}(\zeta_1\, \zeta_2)$. Thus, the set

$$\mathcal{R}_{\mathcal{E}} \equiv \bigcup_{\zeta \in \mathcal{U}} \mathcal{R}(\zeta) , \tag{18}$$

is a subgroup of $\mathcal{U}$. Moreover, unless the sequence of polynomials reduces to one monomial, there are only finitely many $\zeta \in \mathcal{U}$ such that $\mathcal{R}(\zeta)$ is nonempty. In this case $\mathcal{R}_{\mathcal{E}}$ has finite order, hence it coincides with one of the groups $\mathcal{U}_n$. On the other hand given any $\omega \in \mathcal{R}_{\mathcal{E}}$, we define

$$\zeta(\omega) = \{\zeta : \omega \in \mathcal{R}(\zeta)\} . \tag{19}$$

[2] From the definition (17), it is clear that this ζ is unique.

Now we define the set

$$\mathcal{R}_{\mathcal{A}} \equiv \{\omega \in \mathcal{U} : p_i(\omega\, y) = \omega\, p_i(y),\ i = 1, \ldots, m\} . \tag{20}$$

It can be seen that $\mathcal{R}_{\mathcal{A}}$ is a subgroup of the group of roots of unity $\mathcal{U}$. Moreover, $\omega \in \mathcal{R}_{\mathcal{A}}$ iff $\zeta(\omega) = \omega^2$, i.e. iff $\omega \in \mathcal{R}(\omega^2)$, so that $\mathcal{R}_{\mathcal{A}}$ is a subgroup of $\mathcal{R}_{\mathcal{E}}$. Note that $\mathcal{R}_{\mathcal{A}}$ has finite order for any nonempty sequence of (nonlinear) polynomials.

Finally we define the subgroup $\mathcal{N}$ of $\mathcal{R}_{\mathcal{E}}$ by

$$\mathcal{N} \equiv \{\omega \in \mathbb{C} : \omega\, p_i(\omega\, y) = \zeta\, p_i(y),\ \text{for some } \zeta \in \mathcal{R}_{\mathcal{A}}\} = \bigcup_{\zeta \in \mathcal{R}_{\mathcal{A}}} \mathcal{R}(\zeta) , \tag{21}$$

and the subgroup of $\mathcal{R}_{\mathcal{A}}$ defined by

$$\mathcal{N}' \equiv \{\zeta \in \mathcal{R}_{\mathcal{A}} : \mathcal{R}(\zeta) \neq \emptyset\} . \tag{22}$$

These groups have the ordering

$$\mathcal{N}' \subseteq \mathcal{R}_{\mathcal{A}} \subseteq \mathcal{N} \subseteq \mathcal{R}_{\mathcal{E}} \subseteq \mathcal{U} . \tag{23}$$

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[2]not quite sure about defining $\zeta(\omega)$ as a set, that is formally different from its only element

Example 2.1. Given the three normal polynomials

$$p_1(y) = y^5 , \quad p_2(y) = y^{13} + a\, y^5 , \quad p_3(y) = y^{21} ,$$

then if $a \neq 0$ we find that $\mathcal{R}(\zeta)$ is nonempty only for $\zeta \in \mathcal{U}_4$ and that

$$\mathcal{N}' = \mathcal{R}_\mathcal{A} = \mathcal{U}_4 \subset \mathcal{N} = \mathcal{R}_\mathcal{E} = \mathcal{U}_8 .$$

On the other hand, if $a = 0$ then

$$\mathcal{N}' = \mathcal{R}_\mathcal{A} = \mathcal{U}_4 \subset \mathcal{N} = \mathcal{U}_8 \subset \mathcal{R}_\mathcal{E} = \mathcal{U}_{16} .$$

3 Symmetric Automorphisms

In this section we investigate the structure of cyclically reduced words that represent polynomial automorphisms possessing nontrivial symmetries. We will show that if g is a nontrivial polynomial automorphism with nontrivial symmetries, then g is conjugate to a map of the form $s\, H^q$. In this form, s is a finite order, affine-elementary symmetry and H is a cyclically reduced symmetry. This decomposition of g gives rise to a subgroup of $\mathrm{Sym}(g)$ isomorphic to $\mathbb{Z} \times \mathbb{Z}_n$ where n is the order of s. Similar subgroups have been found for the particular case of polynomial mappings of generalized standard form [13].

Using Thm. 5 we can, with no loss of generality, assume that g is in Hénon normal form.

Theorem 7. *Suppose that $g = h_m\, h_{m-1} \cdots h_1$ is a polynomial automorphism in Hénon normal form with a nontrivial symmetry in $\mathcal{G}$. Then there exist diagonal linear transformations $s, \tilde{s}$ such that s has finite order and*

$$g = s^j\, (\tilde{g})^q , \quad \text{where } \tilde{g} = \tilde{s}\, h_r \cdots h_1 ; \tag{24}$$

where $s, \tilde{g}$ are commuting symmetries of g, $m = qr$, and either $s \neq id$ or $q \neq 1$.

Proof. By assumption there is a nontrivial symmetry, f, and by the condition (2), since g is cyclically reduced, f must be either an affine-elementary map or a nontrivial cyclically reduced word. The set $\{g^j\, f^k : j, k \in \mathbb{Z}\}$ is a subgroup of $\mathrm{Sym}(g)$. By replacing f with some convenient element in that subgroup if necessary and using Thm. 4, we may assume that $f = s_m\, h_k \cdots h_1$, with s_m a diagonal affine map, and $0 \leq k < m$. In this case the relation $f\, g\, f^{-1} = g$ becomes

$$s_m\, h_k \cdots h_1\, h_m \cdots h_{k+1}\, s_m^{-1} = h_m \cdots h_1.$$

Since $h_i = t\, e_i$, Thm. 3 then implies the existence of diagonal affine maps s_i, $i = 0 \ldots m-1$, with $s_0 = s_m$ such that

$$h_{i+k} = s_i^{-1}\, h_i\, s_{i-1}, \tag{25}$$

where the indices should be understood mod m.

Let $r = \gcd(m, k)$ the greatest common divisor of m and k, and define integers q, p such that $m = qr$, $k = pr$. In this case there exist integers j and a such that $r = jk - am$ where

we may assume $j, a > 0$ [17]. It then follows that $r \equiv jk \bmod m$, so that $h_{r+i} = h_{jk+i}$. Iterating (25) we obtain

$$h_{i+r} = s^{-1}_{i+(j-1)k} \cdots s^{-1}_{i+k} s^{-1}_i \, (h_i) \, s_{i-1} s_{i-1+k} \cdots s_{i-1+(j-1)k} \, .$$

Defining $\tilde{g}_n = h_{nr} h_{nr-1} \cdots h_{(n-1)r+1}$, we then obtain

$$\tilde{g}_{n+1} = \tilde{s}^{-1}_n \, \tilde{g}_n \, \tilde{s}_{n-1} \, , \quad \text{for } n = 1, \ldots, q-1 \, , \tag{26}$$

where $\tilde{s}_n = s_{nr} s_{nr+k} \cdots s_{nr+(j-1)k}$. Since $m = qr$, we can use induction on (26) to obtain

$$g = \tilde{s}^{-1}_{q-1} \cdots \tilde{s}^{-1}_1 \tilde{s}^{-1}_0 \, \tilde{g}^q \tag{27}$$

where

$$\tilde{g} = \tilde{s}_0 \, h_r \cdots h_1 = \tilde{s}_0 \, \tilde{g}_1 \, . \tag{28}$$

The leading affine-elementary map in (27) is actually the j-th power of a simpler map since $r \equiv jk \bmod m$ implies that $\tilde{s}_n = s_{njk} s_{(nj+1)k} \cdots s_{((n+1)j-1)k}$. Using this we regroup the q groups of j terms as j groups of q to obtain

$$\begin{aligned}
\tilde{s}_0 \tilde{s}_1 \cdots \tilde{s}_{q-1} &= \big(s_0 s_k \cdots s_{(j-1)k}\big)\big(s_{jk} \cdots s_{(2j-1)k}\big) \quad \cdots \quad \big(s_{(q-1)jk} \cdots s_{(jq-1)k}\big) \\
&= \big(s_0 s_k \cdots s_{(q-1)k}\big)\big(s_{qk} \cdots s_{(2q-1)k}\big) \cdots \big(s_{(j-1)qk} \cdots s_{(jq-1)k}\big) \\
&= \big(s_0 s_k \cdots s_{(q-1)k}\big)^j
\end{aligned}$$

since $qk \equiv 0 \bmod m$. Thus we have shown that whenever g has a symmetry it has the form (24) with

$$s \equiv s^{-1}_{(q-1)k} \cdots s^{-1}_k s^{-1}_0.$$

We still have to prove that $s \in \mathrm{Sym}(g)$. Using (25) and the fact that $h_{m+i} = h_i$ allows us to obtain

$$h_i = s^{-1}_{m+i-k} \cdots s^{-1}_{m+i-qk} \, (h_{m+i-qk}) \, s_{m+i-1-qk} \cdots s_{m+i-1-k}.$$

As $qk \equiv 0 \bmod m$ the above relations imply

$$s_{m+r-qk} \cdots s_{m+r-k} \, (h_r \cdots h_1) = (h_r \cdots h_1) \, s_{m-qk} \cdots s_{m-k},$$

which readily yields $s\,\tilde{g} = \tilde{g}\,s$. To see that s must be of finite order, note that since g is assumed to be in normal Hénon form, the relation

$$s \, h_m \cdots h_1 \, s^{-1} = h_m \cdots h_1$$

means that s is a diagonal linear transformation

$$s : (x, y) \rightarrow (a_0 x, a_1 y),$$

that conjugates a normal form to itself. Then, as in Lem. 6, the scaling factors a_0, a_1 must be roots of unity, so that s has finite order. Similarly since each of the s_i are diagonal, so is $\tilde{s}_0$.

To finish we show that if f is a nontrivial symmetry then either $s \neq id$ or $q \neq 1$ To see this we observe that (26) gives

$$\begin{aligned} f &= s_m \, h_k \, \cdots \, h_1 \,, \\ &= s_m (\, \tilde{s}_{p-1}^{-1} \, \cdots \, \tilde{s}_0^{-1}) \, (\tilde{s}_0 \, h_r \, \cdots \, h_1)^p \,, \\ &= s^a \, \tilde{g}^p \,, \end{aligned}$$

where we used $k = pr$ and that $jk \equiv r \mod m$. Therefore if $q = 1$ then $m = r = k$ so that $j = p = 1$, so that when $s = id$ then $f = \tilde{g} = g$ is a trivial symmetry. $\qquad \square$

Note that since s has finite order and is a diagonal linear map, then if it is real it must be an involution.

From this theorem we see that every polynomial automorphism that possesses nontrivial symmetries and is not the q-fold iteration of a nontrivial automorphism must have nontrivial symmetries conjugate to affine-elementary maps. In the case of a map given in Hénon normal form it is not difficult to establish conditions under which the group of such symmetries is nontrivial.

Proposition 8. *If s is an affine-elementary symmetry of the Hénon normal form map $g = h_m \, \cdots \, h_1$, then*

$$s(x, y) = \left(\frac{\zeta(\omega)}{\omega} x, \omega y \right), \quad \omega \in \mathcal{U}_k$$

where $\mathcal{U}_k = \mathcal{R}_\mathcal{E}$ if m is even, and $\mathcal{U}_k = \mathcal{R}_\mathcal{A}$ if m is odd. Thus the set of affine elementary symmetries is a cyclic group isomorphic to $\mathcal{U}_k$.

Proof. According to Thm. 3, $s \in \mathcal{S}$ is a symmetry if and only if there exist maps $s_i \in \hat{\mathcal{S}}$, $i = 0, \ldots, m$, $s_0 = s_m = s$, such that

$$s_i \, h_i \, s_{i-1}^{-1} = h_i. \tag{29}$$

Moreover, given that g is in normal form, each of the maps s_i must be of the form $s_i(x, y) = (a_i x, a_{i+1} y)$. In that case (29) translates into the conditions

$$\begin{aligned} a_{i-1} &= a_{i+1}, \quad a_0 = a_m \\ p_i(a_i x) &= a_{i+1} p_i(x). \end{aligned}$$

When m is even, we must have $a_{2k+1} = \omega$ and $a_{2k} = \zeta(\omega)/\omega$ for $\omega \in \mathcal{R}_\mathcal{E}$ and $\zeta(\omega)$ is defined by (19). Thus $s = s_0$ has the promised form. Since $\zeta(\omega^2) = (\zeta(\omega))^2$, the set of symmetries $s \in \mathcal{S}$ is a cyclic group isomorphic to $\mathcal{R}_\mathcal{E}$.

In a similar way if m is odd, all a_i must be equal to some $\omega \in \mathcal{R}_\mathcal{A}$, (20), so that $s(x, y) = (\omega x, \omega y)$. Moreover since $\zeta(\omega) = \omega^2$ for $\omega \in \mathcal{R}_\mathcal{A}$, the symmetries $s \in \mathcal{S}$ are of the promised form, and they form a cyclic group isomorphic to $\mathcal{R}_\mathcal{A}$. $\qquad \square$

Finally, as a corollary to these results we can prove a more complete statement of Thm. 1.

Corollary 9. *Suppose g is a polynomial automorphism of the plane that possesses nontrivial symmetries. Then g is conjugate to a map of the form*

$$s(H)^q ,$$

where $H = h_n h_{n-1} \ldots h_1$ is a composition of normal generalized Hénon maps (1), $s = (\frac{\zeta(\omega)}{\omega} x, \omega y)$, where $\omega \in \mathcal{R_E}$ if nq is even and $\omega \in \mathcal{R_A}$ if nq is odd. The normal form has commuting symmetries s and H, and either $\omega \neq 1$ or $q \neq 1$.

Proof. According to Thm. 7, g is conjugate to $s^j(\tilde{s}H)^q$. Using Thm. 6, we can conjugate this map with a new diagonal-affine transformation to normalize the map $\tilde{s}H$. The conjugacy commutes with s^j since both are diagonal, and according to Prop. 8, s^j has the promised form. □

4 Reversible automorphisms

In [15] we described conjugacy classes for polynomial automorphisms that are reversible by involutions. Although the involution condition appears in a natural way in many cases and it was originally one of the ingredients in the definition of reversible systems [5], this requirement can be relaxed. Indeed, many of the features still hold in the more general case (see, e.g., [8] for further discussion). Moreover, all maps with an involutory reversor are reversible in the more general sense. Finally, it is certainly true that there exist reversible maps that do not possess any involution as a reversor; for example [9]. We begin by showing that the general reversor in $\mathcal{G}$ is conjugate to one that is affine or elementary. Then we will show that every reversor in $\mathcal{G}$ is of finite order. Finally we prove the main theorem.

Affine and Elementary Reversors

Lemma 10. *If $g \in \mathcal{G}$ is nontrivial and reversible, then it is conjugate to an automorphism $\tilde{g}$ that has an elementary or an affine reversor. If the semilength of g is odd, then $\tilde{g}$ has an affine reversor.*

Proof. Following (12), we may assume that $g = t\,e_m \cdots t\,e_1$ is written as a cyclically reduced word. Similarly let R be the reduced word for a reversing symmetry of g. Since both g and $g^{-1} = R^{-1}\,g\,R$ are cyclically reduced, then unless R is in $\mathcal{A} \cup \mathcal{E}$, it cannot be a nontrivial cyclically reduced map. Recall that if R is a reversor for g so are the maps $R\,g^j$ and that they have the same order as R if this order is even. Thus without loss of generality, we can assume that R is shorter than g; otherwise as in the proof of Thm. 4, there is an $\tilde{R} = R\,g^j$ for some $j \in \mathbb{Z}$, such that $\tilde{R}$ shorter than g and is also a reversor. Moreover, by replacing R with $R\,g$ if necessary, we can assume that

$$R = s_0^{-1}\,e_k\,t \cdots t\,e_1, \ s_0 \in \hat{\mathcal{S}} , \quad k < m, \ s_0 \in \hat{\mathcal{S}} . \tag{30}$$

Following Thm. 4 and given that $g = R g^{-1} R^{-1}$, there must exist maps $s_i \in \hat{\mathcal{S}}$ such that $t e_{i^*}^{-1} = s_i t e_i s_{i-1}^{-1}$ for $i^* = k - i + 1$ and $i = 1, \ldots, m$, where the indices are taken modulus m. Consider then the map $\hat{g} = f g f^{-1}$, where $f = t e_\nu \cdots t e_1$, and $k = 2\nu$ or $k = 2\nu + 1$. Then a simple calculation shows that $\hat{g}$ has a reversor $\hat{R} = f R f^{-1}$ that is either $s_\nu^{-1} t$ when k is even or $s_\nu^{-1} e_{\nu+1}$ when k is odd. Thus $\hat{g}$ is conjugate to g and has a reversor in $\mathcal{A} \cup \mathcal{E}$.

Now suppose that the semilength of the conjugacy class is odd and that $g = t e_{2k+1} \cdots t e_1$ has an elementary reversor, $R = s_0^{-1} e_1$. Reordering terms we see that the map

$$\hat{g} = t e_{k+1} \cdots t e_2 t e_1 t e_{2k+1} \cdots t e_{k+2}$$

has an affine reversing symmetry $\hat{R}$, conjugate to some reversor in the family $R g^j$. Therefore in the case of conjugacy classes of odd semilength we may always assume affine reversors. $\quad\square$

Remark. Along the same lines of the proof of the previous lemma it can be seen that if R is any reversor of a cyclically reduced, nontrivial automorphism g, then R can be written as $R_0 g^j$ for some $j \in \mathbb{Z}$ and R_0 shorter than g. Since $(R_0 g^j)^{2\mu+1} g^l = R_0^{2\mu+1} g^{j+l}$ it follows that the reversors generated by $R_0 g^j$ form a subset of the family generated by R_0.

We can go further by conjugating with maps in S and replacing g by its inverse if necessary, so that g can be taken to have the form (12) and R_0, the form (30). Then, a short calculation shows that the reversors generated by R_0 are of the form

$$R = R_0^{2\mu+1} g^j = (s_0^{-1} t s_k^{-2\mu} t s_0^{-1}) e_k t \cdots t e_1 g^j, \quad s_0, s_k \in \hat{\mathcal{S}}$$

whenever the index $\mu \geq 0$. For negative μ a similar result follows; however if the reversor has finite order, it is enough to consider only one of these possibilities. Now, since $s_0, s_k \in \hat{\mathcal{S}}$ the term inside the parenthesis in the above expression reduces to an elementary affine map. When $j \geq 0$ no further simplifications are possible, so that the reduced word that represents R has length $2k - 1$, if the length is considered modulus $2m$. For $j < 0$ an additional simplification yields

$$R = (s_0^{-1} t s_k^{-2\mu} t s_0^{-1}) t e_{k+1}^{-1} \cdots e_m^{-1} t g^{j+1},$$

so that, modulus $2m$, the length of the word becomes $2(m - k) + 1$. We can conclude that if any two reversors of a nontrivial, cyclically reduced map belong to a common family, then their modulus $2m$ lengths must coincide or be complementary (i.e. their sum is equal to $2m$).

Finite Order Reversors

We now show that reversors in $\mathcal{G}$ have finite order. Recall that when R is a reversing symmetry for g, each of the maps (5) is also a reversor and that whenever that R has finite order, the order must be even, unless g is an involution.

Theorem 11. *Every polynomial reversor of a nontrivial, polynomial automorphism has finite even order. In the case of real transformations the order is 2 or 4.*

Proof. Following Thm. 5, we may assume that g is in Hénon normal form and from Lem. 10 that R is an affine or an elementary reversing symmetry for g. Now, it is easy to see that if R is affine, $R = s_0^{-1} t$, and if R is elementary, $R = s_0^{-1} e_1$, for some $s_0 \in \hat{\mathcal{S}}$, and $e_1(x,y) = (p_1(y) - \delta_1 x, y)$ a normalized elementary map. The condition $g = R\,g^{-1}\,R^{-1}$ is then equivalent to the existence of diagonal linear maps, $s_i(x,y) = (a_i\,x, a_{i+1}\,y)$, $s_m = s_0$, such that

$$t\,e_{i^*}^{-1} = s_i\,t\,e_i\,s_{i-1}^{-1}\,, \quad \text{where} \begin{cases} i^* = m - i + 1, & R \text{ affine} \\ i^* = m - i + 2, & R \text{ elementary} \end{cases} . \tag{31}$$

Here the indices are understood modulus m. This in turn means that

$$\begin{aligned} \delta_i\,\delta_{i^*} &= \frac{a_{i-1}}{a_{i+1}} = \frac{a_{(i+1)^*}}{a_{(i-1)^*}} \quad , \quad i = 1, \dots, m\,. \\ p_{i^*}(a_i\,y) &= \delta_{i^*}\,a_{i+1}\,p_i\,(y) \end{aligned} \tag{32}$$

Defining $\omega_i = a_i\,a_{i^*}$, then (32) implies

$$\begin{aligned} \omega_i &= \omega_{i^*}\,, \quad \omega_{i-1} = \omega_{i+1} \quad , \quad i = 1, \dots, m\,. \\ p_i(\omega_i\,y) &= \omega_{i-1}\,p_i\,(y) \end{aligned} \tag{33}$$

Thus all the odd ω_i are equal, as are all the even ω_i. Furthermore $\omega_m \equiv \omega_0$. It follows that when R is affine or when the semilength of g is odd, all of the ω_i coincide. Then (33) implies that all ω_i are primitive roots of unity of the same order. The proof is complete upon noting that $R^2(x,y) = (x/\omega_0, y/\omega_1)$ which means that R has order $2n$, where n is the order of ω_i. It can be observed that (33) implies that $2n$ must be a factor of $2\,(l_i\,l_{i-1} - 1)$, for all indices i. Finally we see that R is real only if $\omega_0, \omega_1 = \pm 1$, so that the order of R is 2 or 4. $\qquad \square$

It is not hard to find normal forms for elementary and affine reversors.

Definition (Normalized Elementary Reversor). An elementary map of the form

$$e : (x, y) \to (p(y) - \delta\,x, \epsilon\,y)\,, \qquad p(\epsilon\,y) = \delta\,p(y), \tag{34}$$

with $p(y)$ a normal polynomial and ϵ^2, δ^2 some primitive n^{th}-roots of unity, will be called a *normalized elementary reversor* of order $2n$. Note that $e^2 = (\delta^2 x, \epsilon^2 y)$.

Definition (Normalized Affine Reversor). Given any primitive n^{th}-root of unity ω, the map

$$\tau_\omega : (x, y) \to (\omega\,y, x) \tag{35}$$

is a *normalized affine reversor* of order $2n$. Note that $\tau_\omega^2 = (\omega x, \omega y)$

These normal forms will form the building blocks for the conjugacy classes of reversible automorphisms.

Let us suppose now that the map $g = h_m \cdots h_1$ is in Hénon normal form and that it has either an elementary or an affine reversing symmetry of order $2n$. In that case the proof of Thm. 11 implies there exist some primitive n^{th}-roots of unity, ω_0 and ω_1, that solve (33).

Comparing this with (17), we see that $\omega_1 \in \mathcal{R}(\zeta)$, for $\zeta = \omega_0\,\omega_1$. Since ω_1 generates $\mathcal{U}_n$, and $\mathcal{R}_{\mathcal{E}}$, (18), is a group containing $\mathcal{R}(\zeta)$, this implies that $\mathcal{U}_n \subseteq \mathcal{R}_{\mathcal{E}}$. If in addition the reversor is affine, then $\omega_0 = \omega_1$ so that $\omega_1 \in \mathcal{R}_{\mathcal{A}}$, (20), which implies that $\mathcal{U}_n \subseteq \mathcal{R}_{\mathcal{A}}$.

Reversibility imposes stronger conditions than are apparent in (33) for some cases. This occurs for words with even semilength that have an elementary reversor, and for words with odd semilength (in this case, as we noted in Lem. 10 we can assume that there is an affine reversor). We note that $i^* = i$ for $i = k+1$ if the reversor is affine and $m = 2k+1$, while this identity follows for $i = 1, k+1$ if the reversor is elementary and $m = 2k$. For such indices, (32) implies the existence of some constants $\hat{\epsilon}_i$ and $\hat{\delta}_i$ such that

$$p_i(\hat{\epsilon}_i\,y) = \hat{\delta}_i\,p_i(y), \quad \hat{\epsilon}_i^2 = \omega_i, \quad \hat{\delta}_i^2 = \omega_{i-1}, \tag{36}$$

where the constants ω_i also satisfy (33) for the corresponding indices.

We can use (33) and (36) to construct reversible maps. Let us suppose, for example, that we have k normalized Hénon transformations, $h_1(y), \ldots, h_k(y)$, a normal polynomial $p_{k+1}(y)$, and a nth-root of unity, $\omega \in \mathcal{R}_{\mathcal{A}}(p_1(y), \ldots, p_{k+1}(y))$, such that (36) holds for $i = k+1$, if we set $\omega_k = \omega_{k+1} = \omega$. Then it is possible to choose the coefficient δ_{k+1} and the remaining Hénon transformations, in such a way that the map $g = h_{2k+1} \cdots h_1$ has an affine reversing symmetry of order $2n$. Furthermore the number of possible choices is finite. As similar statements follow in the other cases, we have a way to generate all conjugacy classes for reversible automorphisms. These conditions also enables us to give an explicit description of conjugacy classes for reversible automorphisms, as well as to provide normal forms, as we show next.

Normal Form Theorem

We are now ready to prove the main result, which was given in the introduction as Thm. 2. Given the previous lemmas, we can now restate the result here in more detail:

Theorem 12. *Let be g a nontrivial automorphism that possesses a reversor of order $2n$, and ω any primitive n^{th}-root of unity. Then g is conjugate to a cyclically reduced map of one of the following classes:*

$$\mathbf{R_{AA}} \qquad \tau_\omega^{-1}\,(h_1^{-1}\,\cdots\,h_k^{-1})\,\tau_\omega\,(h_k\,\cdots\,h_1)\,, \qquad \omega \in \mathcal{R}_{\mathcal{A}}(p_1(y), \ldots, p_k(y))$$

$$\mathbf{R_{AE}} \qquad \tau_\omega^{-1}\,(h_1^{-1}\,\cdots\,h_k^{-1})\,e_{k+1}\,(h_k\,\cdots\,h_1)\,, \qquad \omega \in \mathcal{R}_{\mathcal{A}}(p_1(y), \ldots, p_{k+1}(y)),$$

$$\mathbf{R_{EE}} \qquad e_1^{-1}\,(t\,h_2^{-1}\,\cdots\,h_k^{-1})\,e_{k+1}\,(h_k\,\cdots\,h_2\,t)\,, \qquad \omega \in \mathcal{R}_{\mathcal{E}}(p_1(y), \ldots, p_{k+1}(y)),$$

where

- *the maps h_i are normalized Hénon transformations,*

- *τ_ω is the normalized affine reversor (35), and*

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- if (ω_i) is the sequence defined by $\omega_1 = \omega$, $\omega_2 = \zeta(\omega)/\omega$, $\omega_{i+1} = \omega_{i-1}$, then the maps e_1, e_{k+1} are normal elementary reversors (34), with $\epsilon_i^2 = \omega_i$, $\delta_i^2 = \omega_{i-1}$.

Furthermore these normal forms are unique up to finitely many choices.

Conversely, τ_ω is a reversing symmetry for any map having normal form $\mathbf{R_{AA}}$ or $\mathbf{R_{AE}}$ while e_1 is an elementary reversor for any map of the form $\mathbf{R_{EE}}$.

Proof. We consider the conjugacy class of a polynomial automorphism $g = t\,e_m \cdots t\,e_1$, having a reversing symmetry R_0, of order $2n$. We may assume that g is given in Hénon normal form and according to Lem. 10 that R_0 is affine or elementary. Moreover, when m is odd it may be assumed that R_0 is affine. Then R_0 is of the form $s_0^{-1}\,t$ or $s_0^{-1}\,e_1$, for some scaling $s_0 : (x,y) \to (a_0\,x, a_1\,y)$. Throughout the present discussion we continue using the notation introduced in Thm. 11. In particular we know that the polynomials $p_i(y)$ satisfy (33). Moreover if R_0 is affine and $m = 2k+1$ the polynomial $p_{k+1}(y)$ also satisfies (36), while if R_0 is elementary and $m = 2k$, this condition is satisfied by $p_1(y)$ and $p_{k+1}(y)$.

We discuss the case R_0 affine; the case that R_0 is elementary follows in a similar way. Note that if R_0 is affine then, $R_0 = \hat{s}_0\,\tau_\omega^{-1}\,\hat{s}_0^{-1}$, for $\omega = a_0\,a_1$ and some diagonal linear map $\hat{s}_0$. Letting $m = 2k$ or $m = 2k+1$, then (31) implies that

$$g = (s_0^{-1}\,t)\left(e_1^{-1}\,t \cdots t\,e_k^{-1}\,[t\,e_{k+1}^{-1}]\,s_k\,t\,e_k \cdots t\,e_1\right) = R_0\,R_1,$$

where the term in brackets is absent if $m = 2k$. Notice that R_1 is also a reversing symmetry of order $2n$, conjugate to either $s_k\,t$ if m is even, or to $e_{k+1}^{-1}\,s_k = \phi(s_{k+1})\,e_{k+1}$ when m is odd. In the first of these cases we also note that there exists a diagonal linear map $\hat{s}_k$ such that $s_k\,t = \hat{s}_k\,\phi(\tau_\omega)\,\hat{s}_k^{-1}$. In the second case, note that the map $\tilde{e}_{k+1} = e_{k+1}^{-1}\,s_k$ is an elementary reversor of the form (34), except that the polynomial $p_{k+1}(y)$ may not be normalized. Therefore g is $\mathcal{S}$-conjugate to a map of the form

$$\tau_\omega^{-1}\,e_1^{-1}\,t \cdots t\,e_k^{-1}\,t\,[e_{k+1}]\,[\tau_\omega]\,t\,e_k \cdots t\,e_1, \tag{37}$$

where some of the maps $t\,e_i$ have been modified, but only by scalings of their variables, so that the polynomials $p_i(y)$ still have center of mass at 0. Furthermore e_{k+1} is a (not necessarily normalized) elementary reversor of the form (34), with $\epsilon_{k+1}^2 = \delta_{k+1}^2 = \omega$. Finally, the brackets indicate the terms that may be omitted, depending on m odd or even.

We can now replace each of the maps $t\,e_i$, $i = 1, \ldots, k$ with normalized Hénon transformations, as well as the maps $e_i^{-1}\,t$ with the corresponding inverses, by applying Lem. 6 to $f = t\,e_k \cdots t\,e_1$. In this case the conjugating maps turn out to be linear transformations. Note that the diagonal linear maps that commute with t also commute with any τ_ω and that the only effect of the conjugacies we apply on e_{k+1} is to rescale the polynomial $p_{k+1}(y)$. In this way we obtain a map of the form (37), conjugate to g, where each of the terms $t\,e_i$, $i = 1, \ldots, k$, is a normalized Hénon transformation. For the even semilength case this already shows that g is conjugate to a map of the form $\mathbf{R_{AA}}$, for some n^{th}-root of unity $\omega \in \mathcal{R}_{\mathcal{A}}(p_1(y), \ldots, p_k(y))$.

When m is odd, we still have to normalize e_{k+1} to make the leading coefficient of $p_{k+1}(y)$ equal to 1. This can be achieved by choosing some convenient scalings

$$s_i : (x, y) \rightarrow (a_i \, x, a_{i+1} \, y), \quad i = 0, \ldots k, \tag{38}$$

$\phi(s_0) = s_0$, to replace $t \, e_{k+1}$ with $\phi(s_k) \, t \, e_{k+1} \, s_k^{-1}$, and then each of the terms $t \, e_i$ with $s_i \, t \, e_i \, s_{i-1}^{-1}$, while the corresponding $t \, e_i^{-1}$ are replaced by $\phi(s_{i-1}) \, t \, e_i^{-1} \, \phi(s_i^{-1}) = \phi(s_i \, t \, e_i \, s_{i-1}^{-1})^{-1}$. It is not hard to see that appropriate coefficients a_i can be chosen to give the normal form, and that they are unique up to lth-roots of unity, with $l = l_1 \cdots l_{k-1} \, (l_k \, l_{k+1} - 1)$.

We still need to show that we can replace ω in these expressions with any given root of unity of order n, and that the forms thus obtained are uniquely determined up to finitely many possibilities.

Let us consider the case of normal form $\mathbf{R_{AA}}$. Using Thm. 4 we see that the terms $t \, e_i$ can be modified only by scalings of the variables, since we require those terms to stay normal. If we apply to the terms $t \, e_i^{-1}$ the images under the isomorphism ϕ of the transformations we use to modify $t \, e_i$, as we did to obtain the normal forms, the structure of the word is preserved and the parameter ω does not change either. In the more general case we may replace $t \, e_i$ with $t \, \hat{e}_i = s_i \, t \, e_i \, s_{i-1}^{-1}$, s_i given by (38), while for $i = 2, \ldots, k - 1$, $t \, e_i^{-1}$ is replaced by $t \, \hat{e}_i^{-1} = \tilde{s}_{i-1} \, t \, e_i^{-1} \, \tilde{s}_i^{-1}$,

$$\tilde{s}_i : (x, y) \rightarrow (\tilde{a}_{i+1} \, x, \tilde{a}_i \, y), \quad i = 1, \ldots, k - 1.$$

In this case, and if $k \geq 2$, it follows that $t \, (e_k^{-1} \, t \, \tau_\omega)$ must be replaced by $t \, (\hat{e}_k^{-1} \, t \, \tau_{\hat{\omega}}) = \tilde{s}_{k-1} \, t \, e_k^{-1} \, t \, \tau_\omega \, s_k^{-1}$ while $\tau_\omega^{-1} \, e_1^{-1}$ becomes replaced with $\tau_{\hat{\omega}}^{-1} \, \hat{e}_1^{-1} = s_0 \, \tau_\omega^{-1} \, e_1^{-1} \tilde{s}_1^{-1}$. If $k = 1$ we have to replace $\tau_\omega^{-1} \, e_1^{-1} \, t \, \tau_\omega$ with $\tau_{\hat{\omega}}^{-1} \, e_1^{-1} \, t \, \tau_{\hat{\omega}} = s_0 \, \tau_\omega^{-1} \, e_1^{-1} \, t \, \tau_\omega \, s_1^{-1}$.

For the structure of the word to remain unchanged we need

$$\tilde{a}_1 = a_0, \quad \lambda_{i-1} = \lambda_{i+1}, \quad \text{and} \quad p_i(\lambda_i \, y) = \lambda_{i+1} \, p_i(y), \tag{39}$$

where i runs from 1 to k, $\tilde{a}_{k+1}$ is defined to be equal to a_k, and $\lambda_i = \tilde{a}_i/a_i$ for $i = 1, \ldots, k+1$, while λ_0 is just defined to be equal to λ_2. We also note that then $\hat{\omega} = \lambda_1 \, \lambda_2 \, \omega$. Now, for the map $\hat{g}$ obtained in this way, to be in normal form, it is also necessary that $\hat{\omega}$ lies in $\mathcal{R_A} = \mathcal{R_A}(\hat{p}_1(y), \ldots, \hat{p}_k(y)) = \mathcal{R_A}(p_1(y), \ldots, p_k(y))$. It follows that, if we set $\zeta = \lambda_1 \lambda_2 = \hat{\omega}/\omega$, ζ is also an element of the group $\mathcal{R_A}$.

We thus have that the solutions (λ_1, λ_2) of (39), yielding alternative normal forms for g, are of the form $(\lambda, \zeta/\lambda)$, for some $\zeta \in \mathcal{R_A}$ and $\lambda \in \mathcal{R}(\zeta)$. Therefore to obtain all possible normal forms it suffices to consider all $\lambda \in \mathcal{N}$ (21), and set $\lambda_1 = \lambda$, $\lambda_2 = \zeta(\lambda)/\lambda$. The requirement that the polynomials $\hat{p}_i(y)$ have leading coefficients equal to 1 allows to determine the coefficients $a_i, \tilde{a}_i$ up to lth-roots of unity, for $l = l_1 \cdots l_{k-1} \, (l_k - 1)$. If $\mathcal{N} = \mathcal{U}_d$ (23), all possible normal forms arise by taking a_1 as any (ld)th-root of unity, and λ as a_1^l.

We show now that $\hat{\omega}$ can be chosen as any primitive n^{th}-root of unity. Note that the possible $\hat{\omega}$ are of the form $\zeta \, \omega$ for some $\zeta \in \mathcal{N}'$. We know that for any $\omega \in \mathcal{R_A}$, $\mathcal{R}(\omega^2)$ is a nonempty set, since it contains ω. Let us denote by $\mathcal{R}_\mathcal{A}^2$ the subgroup of $\mathcal{R_A}$ that consists of elements of the form ω^2, $\omega \in \mathcal{R_A}$. It then follows that $\mathcal{R}_\mathcal{A}^2$ is a subgroup of $\mathcal{N}'$. Now, given

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that $\mathcal{R}_\mathcal{A} = \mathcal{U}_r$ for some r, it is not difficult to see that $\mathcal{R}_\mathcal{A}^2$ is a maximal subgroup of $\mathcal{R}_\mathcal{A}$ if r is even while $\mathcal{R}_\mathcal{A} = \mathcal{R}_\mathcal{A}^2$, if r is odd. In the last case, and in general whenever $\mathcal{N}' = \mathcal{R}_\mathcal{A}$, we see that g can be written in normal form with ω replaced by any rth-root of unity, and in particular by any primitive n^{th}-root of unity.

However if $\mathcal{R}_\mathcal{A}$ has even order, it is possible that $\mathcal{N}'$ reduces to $\mathcal{R}_\mathcal{A}^2 \neq \mathcal{R}_\mathcal{A}$ and then it is no longer clear that $\hat{\omega}$ can be chosen as an arbitrary n^{th}-root of unity. A short calculation shows that $\hat{\omega}$ can still be taken as any n^{th}-root of unity as long as the number r/n be even. When r/n is odd the only admissible n^{th}-roots of unity are the numbers $\exp \frac{i2\pi\nu}{n}$, with ν odd. In particular all primitive n^{th}-roots of unity are still possible, but $\hat{\omega}$ cannot be taken equal to 1, i.e. g lacks involutory reversing symmetries associated to this normal form. It is still possible that there exist involutory reversors, however corresponding to a different reordering of the terms, in the case that the map has other families of reversing symmetries.

In the case of normal form $\mathbf{R_{AE}}$ condition (39) should hold for $i = 1, \ldots, k+1$ (although $\tilde{a}_{k+1}$ is not necessarily equal to a_k), if λ_{k+2} is defined to be equal to λ_k. We still have $\hat{\omega} = \lambda_1 \lambda_2 \omega$, but we need in addition that

$$\hat{\epsilon}_{k+1} = \lambda_{k+1}\, \epsilon_{k+1}, \quad \hat{\delta}_{k+1} = \lambda_k\, \delta_{k+1}, \quad \hat{\epsilon}_{k+1}^2 = \hat{\delta}_{k+1}^2 = \hat{\omega}.$$

These conditions imply that the solutions (λ_1, λ_2) for (39), yielding normal forms for g must be of the form $\lambda_1 = \lambda_2 = \lambda$ for some $\lambda \in \mathcal{R}_\mathcal{A}$. In other words the set $\mathcal{N}$ coincides with $\mathcal{R}_\mathcal{A}$, while the corresponding $\mathcal{N}'$ equals $\mathcal{R}_\mathcal{A}^2$. The statements about admissible $\hat{\omega}$ then follows, basically unchanged.

We can make analogous considerations for normal form $\mathbf{R_{EE}}$. The possible normal forms are obtained in this case by considering any $\lambda \in \mathcal{R}_\mathcal{E}$. If we set $\lambda_1 = \lambda$ and $\lambda_2 = \zeta(\lambda)/\lambda$, the remaining λ_i become determined by $\lambda_{i+1} = \lambda_{i-1}$. Once λ is fixed the requirement that all polynomials $p_i(y)$ be normal, determine the coefficients $a_i, \tilde{a}_i$ up to lth- roots of unity, $l = l_1 \cdots l_{k-1} (l_k\, l_{k+1} - 1)$. Additionally we have

$$\hat{\epsilon}_i = \lambda_i\, \epsilon_i, \quad \hat{\delta}_i = \lambda_{i-1}\, \delta_i, \quad \hat{\epsilon}_i^2 = \hat{\omega}_i, \quad \hat{\delta}_i^2 = \hat{\omega}_{i-1}, \quad i = 1, k+1.$$

In that case $\hat{\omega}_1 = \lambda_1^2\, \omega_1$. Therefore the possible $\hat{\omega}_1$ are of the form $\zeta\, \omega_1$, with $\zeta \in \mathcal{R}_\mathcal{E}^2$. Respect to which $\hat{\omega}$ are allowed, there follows similar conclusions to those obtained in the case of normal form $\mathbf{R_{AA}}$, after replacing $\mathcal{N}$ with $\mathcal{R}_\mathcal{E}$ and $\mathcal{N}'$ with $\mathcal{R}_\mathcal{E}^2$.

Finally, the last assertion of the Theorem follows by direct calculation. $\qquad \square$

Corollary 13. *A map $f \in \mathcal{A} \cup \mathcal{E} \setminus \mathcal{S}$ is a reversing symmetry for some nontrivial, cyclically reduced automorphism $g \in \mathcal{G}$ if and only if f is $\mathcal{S}$-conjugate to either a normalized affine reversor or to a normalized elementary reversor.*

Corollary 14. *Every polynomial involution is conjugate to either of the normal involutions, i) $(x, y) \rightarrow (p(y) - x, y)$, $p(y)$ a normal polynomial, ii) $(x, y) \rightarrow (p(y) - x, -y)$, $p(y)$ normal and even, iii) $(x, y) \rightarrow (p(y) + x, -y)$, $p(y)$ normal and odd, or iv) $(x, y) \rightarrow (y, x)$.*

Proof. Every involution is a reversing symmetry for some nontrivial automorphism. $\qquad \square$

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Corollary 15. *An elementary, nonaffine, map*

$$e : (x, y) \rightarrow (p(y) - \delta\, x, \epsilon\, y + \eta) \tag{40}$$

is a reversing symmetry of a nontrivial, cyclically reduced map in $\mathcal{G}$ if and only if it has finite even order $2n$, $e^2 \in \mathcal{S}$, and ϵ^2, δ^2 are primitive n^{th}-roots of unity.

On the other hand an affine, nonelementary, map

$$a : (x, y) \rightarrow \hat{a}(x, y) + (\xi, \eta), \tag{41}$$

$\hat{a}$ a linear transformation, is a reversing symmetry of a nontrivial, cyclically reduced map in $\mathcal{G}$ if and only if it has finite even order $2n$, and $a^2 \in \mathcal{S}$.

Proof. That these conditions are necessary follows easily from Cor. 13. To see the sufficiency we prove that elementary (resp. affine) maps satisfying such conditions are $\mathcal{S}$-conjugate to normalized elementary reversors (resp. normalized affine reversors).

Let us consider the case of an elementary map (40) of order $2n$, such that ϵ^2, δ^2 are primitive n^{th}-roots of unity and e^2 is an affine transformation. It is not difficult to see that the condition $e^{2n} = id$ implies that if $\eta \neq 0$ then $\epsilon \neq 1$. This observation allows to prove that e can always be conjugated to an elementary map (40), having $\eta = 0$. Moreover, the conjugating maps can be chosen in $C_{\mathcal{S}}(t)$.

Next, we see that when $\eta = 0$ the conditions $e^2 \in \mathcal{S}$, $e^{2n} = id$ reduce to the fact that $\epsilon^{2n} = \delta^{2n} = 1$, plus the existence of some constants A and B such that,

$$p(\epsilon\, y) - \delta\, p(y) = A\, y + B,$$
$$A(\delta^{2n-2} + \delta^{2n-4}\, \epsilon^2 + \cdots + \epsilon^{2n-2}) = 0,$$
$$B(\delta^{2n-2} + \delta^{2n-4} + \cdots + 1) = 0.$$

Straightforward calculations then show that it is possible to choose maps in $\mathcal{S}$ that conjugate e to a normalized elementary reversor. It may be interesting to note that if $\epsilon^2 \neq \delta^2$ the condition $e^{2n} = id$ may be omitted and still it can be granted that e is an elementary reversor of order $2n$.

The case of affine reversors can be worked out in a similar way. It is convenient to prove first that, under the given conditions, an affine map is $\mathcal{S}$-conjugate to its linear part. To obtain this result it is useful to treat the cases $n = 1$ and $n \geq 2$ separately. If $n \geq 2$, the conditions $a^{2n} = id$, $a^2 \in \mathcal{S}$ (a nonelementary) are equivalent to $\hat{a}^2 = \omega\, (id)$, where $\omega = -\det \hat{a}$ is a root of unity of order n. When $n = 1$ we need, in addition, that the vector (ξ, η) be an eigenvector of $\hat{a}$ with associated eigenvalue -1. Finally, it is not difficult to check that a linear, nonelementary map $\hat{a}$ of order $2n$, that satisfies the hypothesis $\hat{a}^2 \in \mathcal{S}$, is $\mathcal{S}$-conjugate to τ_ω for $\omega = -\det \hat{a}$. $\square$

Corollary 16. *A polynomial automorphism is reversible by involutions in $\mathcal{G}$ if and only if it is conjugate to any of the normal forms $\mathbf{R_{AA}}$, $\mathbf{R_{AE}}$, or $\mathbf{R_{EE}}$, with $\omega = 1$, so that e_1 and e_{k+1} are normal elementary involutions and h_i, $i = 1, \ldots, k$, are arbitrary normal Hénon transformations.*

Corollary 17. *A real polynomial automorphism has real reversors in $\mathcal{G}$ if and only if it is reversible by involutions, so that it is conjugate to one of the (real) normal forms $\mathbf{R_{AA}}$, $\mathbf{R_{AE}}$, or $\mathbf{R_{EE}}$, with $\omega = 1$, or if it has a reversing symmetry of order 4 and is conjugate to a normal form map $\mathbf{R_{AA}}$, with $\omega = -1$, so that the maps h_i represent normal, real, Hénon transformations, whose respective polynomials $p_i(y)$ are odd.*

Proof. We noted earlier that the only possible real, reversing symmetries are of order 2 or 4. However there are no elementary, real, reversing symmetries of order 4, since this would imply that e is of the form (40), with ϵ^2, δ^2 primitive square roots of unity. Therefore the only possible normal form for a real map with a real reversor of order 4 is $\mathbf{R_{AA}}$, with $\omega = -1$. $\qquad \square$

5 Examples

In this section we illustrate some of the concepts and results of §3 and §4. We present several examples to illustrate the general theory; we do not assert that these examples are new. Examples of maps with nontrivial symmetry groups are well known [18, 19]. Examples of maps with noninvolutory reversors have been presented before; for example, Lamb has constructed examples of "modified Townsville" maps with a reversor of order $4n + 2$ for any n—these maps also have involutory reversors [9]. In addition, Roberts and Baake have shown that "generalized standard maps", a particular case of semilength 2 maps, can have 4th order reversors [13].

For the case of semilength one, i.e. a single generalized Hénon map, (1), the symmetric and reversible maps are easily found. As shown in Prop. 8, there is an affine-elementary symmetry $S(x, y) = (\omega x, \omega y)$ for any $\omega \in \mathcal{R_A} = \{\omega : p(\omega y) = \omega p(y)\}$. Since we can take ω to generate $\mathcal{R_A}$, and S and h commute, $\mathrm{Sym}(h) = \langle h \rangle \times \langle S \rangle \simeq \mathbb{Z} \times \mathcal{R_A}$. For example when p is odd, the Hénon map has the reflection symmetry $S(x, y) = (-x, -y)$.

The reversible cases of (1) can be obtained using (32). When $\delta \neq \pm 1$, there are no reversors, so that $\mathrm{Rev}(h) = \mathrm{Sym}(h)$. If $\delta = 1$, then h has the involutory reversor t, which commutes with S so that $\mathrm{Rev}(h) = \mathrm{Sym}(h) \rtimes \langle t \rangle \simeq (\mathbb{Z} \rtimes \mathbb{Z}_2) \times \mathcal{R_A}$. The case $\delta = -1$ is reversible with reversor $R = (\epsilon y, \epsilon x)$, providing $\epsilon^2 \in \mathcal{R_A}$ and $p(\epsilon y) = -\epsilon p(y)$. There are two possibilities: if the order of $\mathcal{R_A}$ is odd, then there is an involutory reversor $R(x, y) = -(y, x)$, and the group as the same structure as the case $\delta = 1$ (for the real case this is the same as Table 5 of [13]). Otherwise the reversors are complex and noninvolutory. There is an ϵ such that the group $\langle R \rangle \simeq \langle S \rangle \times \mathbb{Z}_2$ gives all affine-elementary symmetries and reversors. Thus we can write $\mathrm{Rev}(h) = \langle h \rangle \rtimes \langle R \rangle \simeq \mathbb{Z} \rtimes (\mathcal{R_A} \times \mathbb{Z}_2)$.

Similarly the symmetries for the semilength two case, $g = h_2 \, h_1$, are also easily found; the results are given in Table 1. There are two possible forms $\mathbf{S_E}$ and $\mathbf{S_A}$, corresponding to the groups $\mathcal{R_E}$ and $\mathcal{R_A}$, respectively. For example in case $\mathbf{S_E}$, there is a symmetry $s(x, y) = (\zeta(\omega)x/\omega, \omega y)$ for any $\omega \in \mathcal{R_E}$. If $\mathcal{R_A}$ is trivial or $\delta_1 \neq \delta_2$, then there are no other nontrivial symmetries, so that $\mathrm{Sym}(g) = \mathbb{Z} \times \mathcal{R_E}$. However, when $\delta_1 = \delta_2$, then there can be additional nonaffine symmetries corresponding to case $\mathbf{S_A}$ providing the polynomials p_1 and p_2 are related by the scaling shown in the table.

According to Thm. 12, there are also two possible reversible cases of semilength two, corresponding to normal forms $\mathbf{R_{AA}}$ and $\mathbf{R_{EE}}$. The conditions for the existence of these can by found by using (32); they are also given in Table 1. Thus, for example, when there is a normal form $\mathbf{R_{AA}}$, the polynomials in h_1 and h_2 must be identical up to a scaling. Moreover, if there are noninvolutory reversors, then the group $\mathcal{R}_{\mathcal{A}}$ must be nontrivial, which implies that $p_i(y) = y\, q_i(y)$ for some polynomials $q_i(y)$ such that the degrees of their nonzero terms are not coprime.

Whenever a reversible map has a noninvolutory reversor, then it also has nontrivial symmetries. For example, for the case $\mathbf{R_{AA}}$ in Table 1, a noninvolutory reversor corresponds to $\omega \in \mathcal{R}_{\mathcal{A}} \setminus \{1\}$. In this case $\tau_\omega^2 = s$ is a symmetry, since $\omega \in \mathcal{R}_{\mathcal{A}} \subset \mathcal{R}_{\mathcal{E}}$. If in addition $\delta_1 = \delta_2 = 1$, then $e^{-1} = e$, and the map has a "square root", and consequently a symmetry of the form $\tilde{s}h$.

Case	Normal Form	Symmetries	Conditions on δ_i	Conditions on $p_i(y)$
$\omega \in \mathcal{R}_{\mathcal{E}}(p_1(y), p_2(y))$				
$\mathbf{S_E}$	$h_2\, h_1$	s	$arbitrary$	
$\mathbf{R_{EE}}$	$\hat{e}_1^{-1}\, t\, \hat{e}_2\, t$	$\hat{e}_1,\ \hat{e}_2$	$\delta_1^2 = \delta_2^2 = 1$	$p_1(\hat{\epsilon}_1\, y) = \delta_1\, \hat{\epsilon}_2\, p_1(y),\ \ \hat{\epsilon}_1^2 = \omega$ $p_2(\hat{\epsilon}_2\, y) = \delta_2\, \hat{\epsilon}_1\, p_2(y)$
$\omega \in \mathcal{R}_{\mathcal{A}}(p_1(y), p_2(y))$				
$\mathbf{S_A}$	$s(\tilde{s}h)^2$	$s,\ \tilde{s}h$	$\delta_1 = \delta_2$	$c\, p_2(cy) = \omega p_1(y)$
$\mathbf{R_{AA}}$	$\tau_\omega^{-1}\, h^{-1}\, \tau_\omega h$	τ_ω	$\delta_1\, \delta_2 = 1$	$c\, p_2(c\, y) = \delta_2\, \omega\, p_1(y)$

Table 1: Conditions for a semilength-two Hénon normal form map, $g = h_2\, h_1$, to have symmetries or be reversible. Here the symmetry $s(x, y) = (\frac{\zeta(\omega)}{\omega} x, \omega y)$ is order n, and the reversors τ_ω and e_i are order $2n$, where n is the order of ω.

Example 5.1. As an example consider the composition of two cubic Hénon maps, $g = h_2 h_1$ where

$$h_1(x, y) = \left(y, y^3 + y - \delta_1 x\right), \quad h_2(x, y) = \left(y, y^3 - y - \delta_2 x\right), \tag{42}$$

For (p_1, p_2),

$$\mathcal{N}' = \mathcal{R}_{\mathcal{E}}^2 = \{1\} \subset \mathcal{R}_{\mathcal{A}} = \mathcal{R}_{\mathcal{E}} = \mathcal{U}_2\ ;$$

Since $\mathcal{R}_{\mathcal{E}}$ is nontrivial, this system has symmetries of the form $\mathbf{S_E}$; indeed, since $\zeta(\omega) = 1$, the affine transformation $S_1(x, y) = (-x, -y)$ is a symmetry. If $\delta_1 \neq \delta_2$, all symmetries are of the form $s^a\, g^p$ so $\mathrm{Sym}(g) = \langle g \rangle \times \langle S_1 \rangle \simeq \mathbb{Z} \times \mathbb{Z}_2$. If however $\delta_1 = \delta_2 = \delta$, then since $i p_2(iy) = p_1(y)$, there are symmetries of the form $\mathbf{S_A}$. We find that $g = (S_2)^2$ with the

symmetry $S_2 = (-iy, ip_1(y) - i\delta x))$. As the group generated by S_2 is still isomorphic to $\mathbb{Z}$, we have $\mathrm{Sym}(g) = \langle S_2 \rangle \times \langle S_1 \rangle \simeq \mathbb{Z} \times \mathbb{Z}_2$.

There are two possible reversible cases for g. When $\delta_i^2 = 1$, Table 1 shows that g potentially can be put in normal form $\mathbf{R_{EE}}$. The scaling relations imply that $\delta_1 = \delta_2 = \delta = \pm 1$ for this to be the case. Then $R_1 = (p_1(y) - \delta x, \delta y)$ is an involutory reversor, and generates a family (5) of reversors that contains all reversors with this ordering.

When $\delta_1 \delta_2 = 1$, (42) potentially has a reversor with normal form $\mathbf{R_{AA}}$. In this case the scaling relations also require $\delta_1 = \delta_2 = \delta = \pm 1$, and there is a reversor $R_2 = (-i\delta y, ix)$. When $\delta = 1$, R_2 is an involution; however, when $\delta = -1$, it is order four and $R_2^2 = S_1$. In both cases, $\delta = \pm 1$, there is an involutory reversor,R_1,; therefore, every reversor can be written as the composition of a symmetry and R_1; for example $R_2 = S_2 R_1$.

Thus we conclude that there are three distinct cases:

1. $\delta_1 = \delta_2 \in \mathcal{U}_2 \Rightarrow \mathrm{Rev}(g) = (\langle S_2 \rangle \times \langle S_1 \rangle) \rtimes \langle R_1 \rangle$.

2. $\delta_1 = \delta_2 \notin \mathcal{U}_2 \Rightarrow \mathrm{Rev}(g) = \mathrm{Sym}(g) = \langle S_2 \rangle \times \langle S_1 \rangle$.

3. $\delta_1 \neq \delta_2 \Rightarrow \mathrm{Rev}(g) = \mathrm{Sym}(g) = \langle g \rangle \times \langle S_1 \rangle$.

where $\langle S_2 \rangle \simeq \mathbb{Z}$, $\langle S_1 \rangle \simeq \mathbb{Z}_2$ and $\langle R_1 \rangle \simeq \mathbb{Z}_2$.

Example 5.2. Let $g = h_2 h_1$ where

$$h_1(x, y) = \left(y, y^3 + x\right), \qquad h_2(x, y) = \left(y, y^3 - x\right),$$

In this case the associated groups of roots of unity are larger:

$$\mathcal{N}' = \mathcal{R}_\mathcal{A} = \mathcal{U}_2 \subset \mathcal{R}_\mathcal{E}^2 = \mathcal{U}_4 \subset \mathcal{N} = \mathcal{R}_\mathcal{E} = \mathcal{U}_8 \ .$$

and $\zeta(\omega) = \omega^4$. Table 1 shows that the nontrivial symmetries are generated by $S_1 = (\omega^3 x, \omega y)$ with $\omega = e^{i\frac{\pi}{4}}$ a primitive, eighth root of unity, and $\langle S_1 \rangle \simeq \mathcal{U}_8$. Since $\delta_1 \neq \delta_2$ there are no symmetries of the form $\mathbf{S_A}$. Thus $\mathrm{Sym}(g) = \langle g \rangle \times \langle S_1 \rangle$.

Since $\delta_1 \delta_2 \neq 1$, g cannot be written in form $\mathbf{R_{AA}}$; however, it can be written in form $\mathbf{R_{EE}}$, for $\hat{\epsilon}_1^8 = -1$. The reversors are generated by $R_1 = (\hat{\epsilon}_1(y^3 + x), \hat{\epsilon}_1^3 y)$, a sixteenth order reversor. Note that $R_1^2 = S_1$, and that R_1 commutes with S_1. Thus $\mathrm{Rev}(g) = \langle g \rangle \rtimes \langle R_1 \rangle \simeq (\mathbb{Z} \rtimes \mathbb{Z}_2) \times \mathbb{Z}_8$.

Note that $\mathcal{R}_\mathcal{E}^2 = \mathcal{U}_4$, so that ω in the $\mathbf{R_{AA}}$ normal form may be replaced only by primitive 8th-roots of unity. Thus, there are no real normal forms and though g is reversible in the group of complex automorphisms, it lacks real reversors.

Example 5.3. Consider the Hénon transformation

$$h : (x, y) \rightarrow \left(y, y^{21} - y^5 - \delta x\right),$$

where $\delta \neq 0$, and let $g = \tau_\omega^{-1} h^{-1} \tau_\omega h$, $\omega = 1$, so that g is reversible by involutions. We see that

$$\mathcal{R}_\mathcal{A}^2 = \mathcal{U}_2 \subset \mathcal{N}' = \mathcal{R}_\mathcal{A} = \mathcal{U}_4 \subset \mathcal{N} = \mathcal{U}_8 \subset \mathcal{R}_\mathcal{E} = \mathcal{U}_{16} \ .$$

Note that $\mathcal{N}' = \mathcal{R}_\mathcal{A} \neq \mathcal{R}_\mathcal{A}^2$. We can conclude that g has also reversors of order 4 and 8. For example, a normal form associated to $\hat{\omega} = -1$, i.e. to reversors of order 4, is given by

$$\hat{g} = s\, g\, s^{-1} = \tau_{\hat{\omega}}^{-1}\, \hat{h}^{-1}\, \tau_{\hat{\omega}}\, \hat{h},$$

where $s(x, y) = (x\, \exp(\frac{7i\pi}{8}), y\, \exp(\frac{3i\pi}{8}))$, and $\hat{h}(x, y) = (y, y^{21} - y^5 + \delta\, x)$. However the conjugating map s cannot be chosen in the group of real automorphisms. This means that g lacks real reversors of order 4, associated to the current ordering of the terms. It turns out that this is always the case, i.e. a real map cannot have real reversors of order 2 and 4, attached to the same ordering of the terms.

Finally, taking e.g. $s(x, y) = (x\, \exp(\frac{61i\pi}{80}), y\, \exp(\frac{i\pi}{80}))$, and $\hat{h}(x, y) = (y, y^{21} - y^5\, \exp(\frac{i\pi}{5}) + i\,\delta\, x)$, we obtain a normal form $\hat{g}$ corresponding to $\hat{\omega} = i$, i.e. to a reversing symmetry of order 8.

For longer words the degrees of the terms can be useful as an indication of which terms may be centers of symmetry, since the normal forms in Thm. 12 have polydegrees that are symmetric about the centers, and the polydegree of a reduced word is a conjugacy invariant. However, explicit conditions analogous to those in Table 1 are more difficult to write.

6   Dynamics

The dynamics of a map is affected in a number of ways by the existence of reversing symmetries. In particular, those orbits that are "symmetric" share many of the typical properties of the orbits of symplectic maps, i.e., their spectral and bifurcation properties. Although our main interest is to discuss plane polynomial diffeomorphisms, we begin with a general discussion, and later focus on the polynomial case. We start by briefly reviewing some of the well known implications of reversibility [20, 14].

Let $\mathcal{O}(x)$ denote the orbit of x under a diffeomorphism g. When $R \in \mathrm{Rev}(g)$, then the symmetry maps orbits into orbits, $R(\mathcal{O}(x)) = \mathcal{O}(R(x))$. Thus orbits either come in symmetric pairs, or are themselves invariant under R. If R is a reversor, then an orbit and its reflection are generated in reverse order.

If $\mathcal{O}(R(x)) = \mathcal{O}(x)$, the orbit is said to be *symmetric* with respect to R. Observe then that the orbit is symmetric respect to any of the reversing symmetries in the subgroup $\langle g, R \rangle$ generated by g and R.

If R is a reversor, then for symmetric orbits forward stability implies backward stability. Moreover, there can be no attractors that are symmetric under R; indeed, if A is a symmetric omega-limit set, then it cannot be asymptotically stable [21].

By contrast, an asymmetric orbit can be attracting, just as long as its symmetric partner is repelling.

If R is a reversor, and x is a point on a symmetric orbit of period n, then the matrix $Dg^n(x)$ is conjugate to its inverse. Thus every multiplier of a periodic symmetric orbit must be accompanied by its reciprocal. When g is real, it follows that eigenvalues other than ± 1 must appear either in pairs (λ, λ^{-1}), with λ real or on the unit circle, or in quadruplets,

$(\lambda, \bar{\lambda}, \lambda^{-1}, \bar{\lambda}^{-1})$. Unlike the symplectic case, 1 and -1 may have odd multiplicity. This situation imposes severe restrictions on the motion of eigenvalues for parameterized families of maps: thus if the multiplicity of 1 or -1 is odd, it must continue to remain so, as long as reversibility is preserved. Therefore, whenever 1 or -1 have odd multiplicity, they should persist as eigenvalues. In the plane this means that for families of reversible, orientation-reversing maps, the spectrum is restricted to the set $\{1, -1\}$. In all the other cases families of reversible maps must preserve orientation.

Though generally reversible maps need not be volume-preserving, reversible polynomial automorphisms are, since their jacobians are necessarily constant. In addition, note that maps with the normal form $\mathbf{R_{AA}}$ are orientation-preserving. Maps with the normal form $\mathbf{R_{AE}}$ or $\mathbf{R_{EE}}$ can either preserve or reverse orientation.

We denote the fixed set of a map R by

$$\mathrm{Fix}(R) \equiv \{x : R(x) = x\} .$$

If $S \in \mathrm{Sym}(g)$ then its fixed set is an invariant set. By contrast, the fixed sets of reversors are not invariant, but contain points on symmetric orbits.

Indeed, as is well-known, to look for symmetric periodic orbits it is enough to restrict the search to the set $\mathrm{Fix}(R) \cup \mathrm{Fix}(g\,R)$ [5, 20, 22, 8]. Therefore if the reversing symmetry has a nontrivial fixed set, it can be used to simplify the computation of periodic points. Indeed, Devaney's original definition of reversibility [5] required that the fixed set of the reversor be a manifold with half the dimension of the phase space. This is the case for maps on the plane that are reversible by orientation-reversing involutions [22]. From this point of view the noninvolutory polynomial reversors we have described are not very interesting, since for each of them the associated symmetric orbits reduce to a single fixed point.

We will now show that this is always the case for order 4 reversing symmetries of the plane. In addition, we will recall the result that in this case the symmetric fixed point is hyperbolic [23].

We start by showing that the fixed set of any order 4 transformation of the plane is a point. This is a well-known result of Brouwer, who showed that finite period transformations of $\mathbb{R}^2$ are topologically equivalent to either a rotation or to the composition of a rotation and a reflection about a line through the origin [24]. Nevertheless, we present an elementary proof of the local nature of the fixed set similar to that given by MacKay for the case of involutions [22] because this proof provides additional information that we find useful later. To complete the description of the fixed set some general results on transformation groups turn out to be necessary.

Lemma 18. *Suppose R is an order 4 diffeomorphism of $\mathbb{R}^2$. Then its fixed set is a point.*

Proof. Assume first that R has a fixed point, without loss of generality, at $(0,0)$. We see that then $f = R^2$ is an orientation-preserving involution, since the jacobian matrix for f at $(0,0)$ equals the square of the corresponding jacobian matrix of R. A simple calculation then shows that $Df(0,0) = \pm I$. Thus we can write

$$f : (x,y) \rightarrow \pm(x,y) + (f_1(x,y), f_2(x,y)),$$

with $f_k(x, y) = o(|(x, y)|)$. Define new variables u and v according to the local diffeomorphism

$$(u, v) = \pm(x, y) + \frac{1}{2}\left(f_1(x, y),\ f_2(x, y)\right)$$

where the sign is chosen in accordance with $Df(0, 0)$. In these local variables the map f reduces to $(u, v) \to \pm(u, v)$. We thus have shown that at each fixed point of R, f is locally conjugate to $\pm id$. Moreover, whenever R has a fixed point, the above condition holds at each of the fixed points of f. This implies that if f is locally the identity, then $f = id$ on its domain, as long as this domain is connected. Therefore, given that R has order 4 and is not an involution, we conclude that $(0, 0)$ is an isolated fixed point and that the (real) normal form for $DR(0, 0)$ is given by

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{43}$$

To prove that R actually has a unique fixed point, we proceed to extend the map R to S^2, by adding a point at infinity and make it a fixed point of R. Now, according to Smith's classical results on transformation groups (see [25]), the fixed set for a transformation of order p^k, (p prime) acting on a n-sphere has the homology-mod p of an r-sphere for some $-1 \leq r \leq n$, where $r = -1$ corresponds to the empty set. We know however that the fixed set of R, acting on the sphere, is nonempty, and that fixed points (at least other than the fixed point at infinity) are isolated. Therefore the fixed set of R must have the mod p homology of S^0, hence it consists exactly of two points. Restricted to the plane we see that R has exactly one fixed point. $\qquad\square$

Notice that whenever a reversing symmetry R for some map g has a single fixed point, this point becomes a symmetric fixed point of g. It was shown by Lamb for the case that R is a rotation by $\pi/4$ that this point cannot be elliptic [23]. Using the previous lemma, it is easy to generalize this to arbitrary order 4 reversors.

Lemma 19. *Suppose g is a reversible map of $\mathbb{R}^2$ with a real reversor R of order 4. Then the associated symmetric fixed point is not elliptic.*

Proof. We may assume that the fixed point is the origin and the jacobian matrix of R at $(0, 0)$ is given by (43). The reversibility condition implies that

$$DR(0, 0)\, Dg(0, 0) = Dg(0, 0)^{-1}\, DR(0, 0).$$

This equation implies that $Dg(0, 0)$ is a symmetric matrix with determinant equal 1. As symmetric real matrices have real eigenvalues, we conclude that the point $(0, 0)$ cannot be elliptic, and that the map is orientation-preserving. $\qquad\square$

To illustrate some of this phenomena, we give two examples.

Example 6.1. Consider a normal form of type $\mathbf{R_{AA}}$:

$$g = \tau_\omega^{-1}\, h^{-1}\, \tau_\omega\, h\ ,$$
$$h(x, y) = (y, y^3 - b\, y - \delta\, x)\ , \quad \omega = -1\ ,$$

so that $\tau_\omega(x, y) = (-y, x)$ is an order 4 reversor for g. If we let $p(y) = y^2 - by$ and assume $b \neq 0$, then $\mathcal{R}_\mathcal{A}(p) = \mathcal{U}_2$, while $\mathcal{N}' = \{1\}$, so the only reversing symmetries for this ordering are order four. Fixed points of this map must satisfy the equations

$$(1 - \delta)x^* = -p(y^*) , \quad (1 - \delta)y^* = p(x^*)$$

where $p(x) = x^3 - bx$. Since the reversor τ_ω has a fixed point at the origin, the origin is always a symmetric fixed point. In general, the stability of a fixed point is determined by

$$\mathrm{Tr}(Dg) = \frac{1}{\delta} \left(p'(y^*)p'(x^*) + \delta^2 + 1 \right)$$

At the origin this becomes $\mathrm{Tr}(Dg(0,0)) = \delta^{-1}(b^2 + \delta^2 + 1)$, which implies, in accord with Lem. 19, that the origin is hyperbolic, since $|\mathrm{Tr}(Dg(0,0))| > 2$, except when $(b, \delta) = (0, \pm 1)$, where it is parabolic.

The remaining 8 fixed points are born together in 4 simultaneous saddle-node bifurcations when

$$b = b_{sn\pm} \equiv \pm 2\sqrt{2}|\delta - 1| .$$

These lines are shown in Fig. 1; inside the cone $b_{sn-} \leq b \leq b_{sn+}$ the map g has only one fixed point. The dynamics of this situation are depicted in Fig. 2 for the case that $\delta = 1.3$.

Outside this cone the map has eight asymmetric fixed points. An example is shown in Fig. 3; for this case four of the fixed points are are elliptic and four are hyperbolic. Note that the four islands surrounding the elliptic fixed points in this figure are mapped into one another by τ_ω. The elliptic fixed points undergo a period-doubling when $\mathrm{Tr}(Dg(x^*, y^*)) = -2$, which corresponds to the curve

$$b^4 - (7 - 13\delta + 7\delta^2)b^2 - 2(2\delta - 1)^2(\delta - 2)^2 = 0$$

This gives the dashed curves shown in Fig. 1. For example if we fix $\delta = 1.3$ and increase b, then period-doubling occurs at $b \approx 1.6792$. The four new period two orbits are stable up to $b \approx 1.7885$, when they too undergo a period-doubling bifurcation. Thus in Fig. 4, there are 4 unstable period-two orbits and 4 more corresponding period four orbits. In this case the stable and unstable manifolds of the saddles intersect, forming a complex trellis.

In contrast to Lem. 19, real maps with order 4 complex reversors, can have elliptic symmetric fixed points.

Example 6.2. Consider for instance the polynomial map, given in Hénon normal form

$$g = h_2 \, h_1 , \quad \text{with} \quad h_k(x, y) = (y, p_k(y) + x) , \tag{44}$$

and assume that $i \in \mathcal{R}_\mathcal{A}(p_1(y), p_2(y))$. According to Table 1, g can be written in normal form $\mathbf{R_{EE}}$ with associated order 4 reversors. Direct calculations show that g is conjugate to the map $\hat{g} = t\,\hat{e}_1^{-1}\,t\,\hat{e}_2$, where

$$\hat{e}_1(x, y) = (\hat{p}_1(y) + i\,x, -i\,y), \quad \text{and} \quad \hat{e}_2(x, y) = (\hat{p}_2(y) - i\,x, i\,y)$$
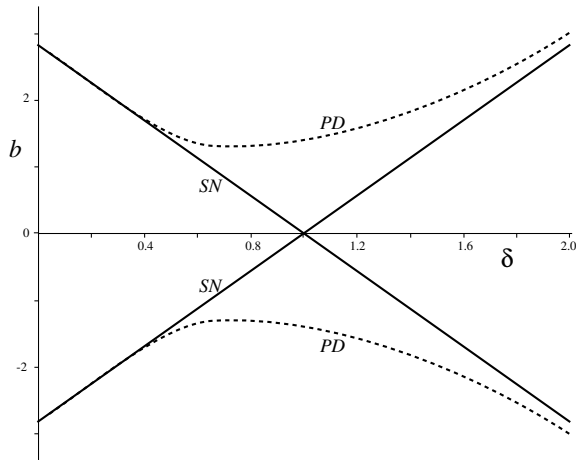
27

Figure 1: Bifurcation curves for the asymmetric fixed points of (42). The 8 asymmetric fixed points exist in the upper and lower quadrants of the cone. They undergo period-doubling bifurcations on the dashed curves.

are elementary normal reversors and the $\hat{p}_k$ are rescalings of p_k. Therefore for some scaling s the map $s^{-1}\hat{e}_2 s$ is an order 4 reversor for g and the origin is a symmetric fixed point. Furthermore $\mathrm{Tr}(Dg(0,0)) = 2 + p_1'(0)\,p_2'(0)$, so that whenever $-4 < p_1'(0)\,p_2'(0) < 0$, the origin is an elliptic point. An example is shown in Fig. 5.

In this example the map g also possesses involutory reversors. In fact this is always the case for orientation-preserving semilength 2 maps with normal form $\mathbf{R_{EE}}$. That is, whenever the map has order 4 reversors there also exist involutory reversors, as can be readily obtained using conditions on Table 1.

7 Conclusions

We have shown that maps in $\mathcal{G}$ that have nontrivial symmetries have a normal form $s(H)^q$ in which either there is a finite order linear symmetry s, or in which the map has a root H. This result is encapsulated in Cor. 9.

Similarly, we have shown that reversors for automorphisms in $\mathcal{G}$ have normal forms that are either affine or elementary. These can be further normalized so that the reversors correspond either to the simple affine map τ_ω (35), or to an elementary reversor of the form (34). These reversors have finite, even order. The case that the order is 2, i.e. involutory reversors, is typical in the sense that the existence of reversors of higher order requires that the polynomials in the map satisfy extra conditions so that one of the groups $\mathcal{R}_\mathcal{A}$ (20) or $\mathcal{R}_\mathcal{E}$ (18) is nontrivial. If a map has real reversors, then they must be order 2 or 4.

Using these, we obtained three possible normal forms for reversible polynomial automorphisms of the plane, Thm. 12. These correspond to having either two affine, two elementary, or one affine and one elementary reversor.
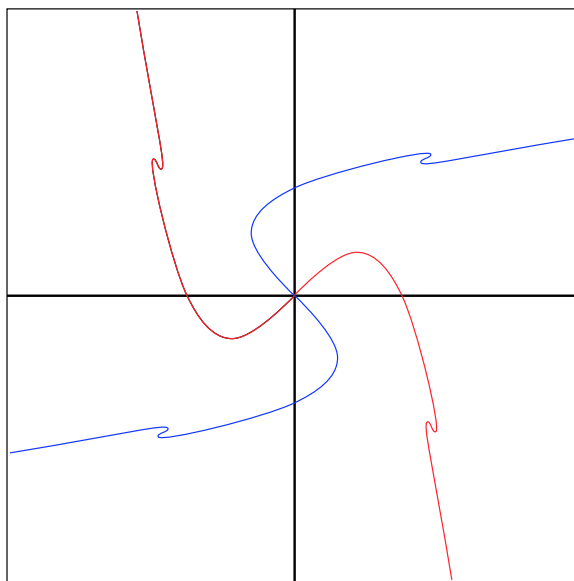
28

Figure 2: Stable and unstable manifolds of the origin for (42) for $(b, \delta) = (0.85, 1.3)$. The domain of the figure is $(-3, 3) \times (-3, 3)$.

It would be interesting to generalize these results to higher dimensional polynomial maps. The main difficulty here is that Jung's decomposition theorem has not been generalized to this case. Nevertheless, one could study the class of polynomial maps generated by affine and elementary maps.
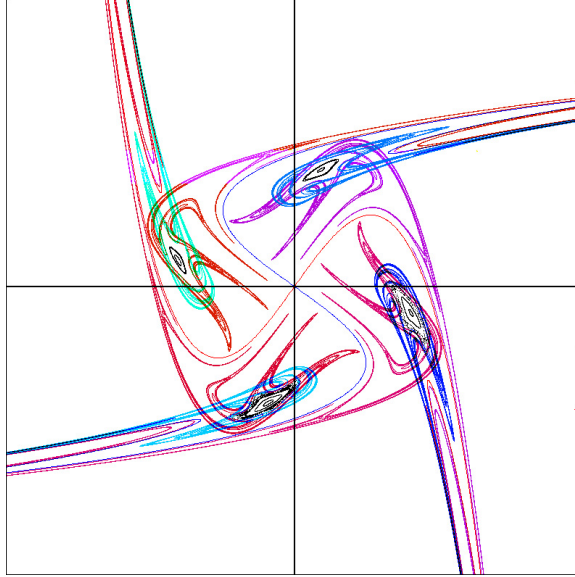
Figure 3: Some orbits of the map (42) for $(b, \delta) = (1.4, 1.3)$. There are elliptic fixed points at $(0.27438, 1.2116)$ and hyperbolic points at $(1.0010, 1.2794)$, as well as the images of these points under τ_ω. The domain is the same as Fig. 2.
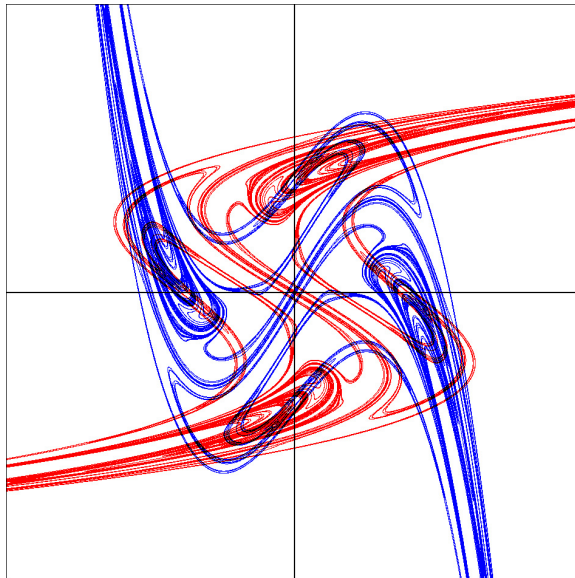

Figure 4: Some stable and unstable manifolds of the map (42) for $(b, \delta) = (1.8, 1.3)$. Here the elliptic points (e.g. at $(0.23390, 1.3607)$) have undergone a period-doubling bifurcation. The domain is the same as Fig. 2.
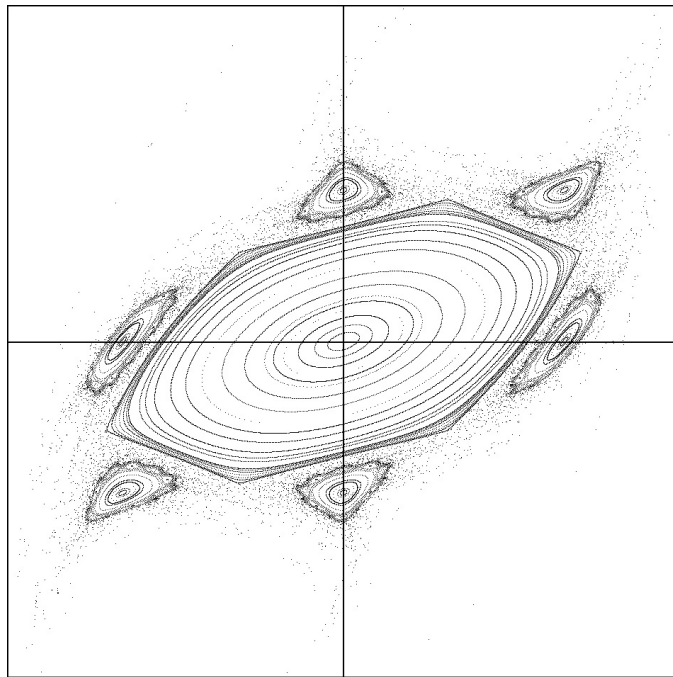
Figure 5: A map of the form (44) with $p_1 = y^5 - 0.5y$ and $p_2 = y^5 + 1.5y$ so that the fixed point at the origin is elliptic. The domain of the figure is $(-1, 1) \times (-1, 1)$.

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