Frequently Asked Questions about Nonlinear Science

J.D. Meiss Sept 2003

[1] About Sci.nonlinear FAQ

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[1.1] What's New?

Fixed lots of broken and outdated links. A few sites seem to be gone, and some new sites appeare d.

To some extent this FAQ is now been superseded by the Dynamical Systems site run by SIAM. See http://www.dynamicalsystems.org There you will find a glossary that contains most of the answ ers in this FAQ plus new ones. There is also a growing software list. You are encouraged to contrib ute to this list, and can do so interactively.

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[2] Basic Theory

[2.1] What is nonlinear?

In geometry, linearity refers to Euclidean objects: lines, planes, (flat) three-dimensional space, etc.--these objects appear the same no matter how we examine them. A nonlinear object, a sphere for example, looks different on different scales--when looked at closely enough it looks like a plane, and from a far enough distance it looks like a point.

In algebra, we define linearity in terms of functions that have the property $f(x+y) = f(x)+f(y)$ and $f(ax) = af(x)$. Nonlinear is defined as the negation of linear. This means that the result f may be out of proportion to the input x or y. The result may be more than linear, as when a diode begins to pass current; or less than linear, as when finite resources limit Malthusian population growth. Thus the fundamental simplifying tools of linear analysis are no longer available: for example, for a linear system, if we have two zeros, $f(x) = 0$ and $f(y) = 0$, then we automatically have a third zero $f(x+y) = 0$ (in fact there are infinitely many zeros as well, since linearity implies that $f(ax+by) = 0$ for any a and b). This is called the principle of superposition--it gives many solutions from a few. For nonlinear systems, each solution must be fought for (generally) with unvarying ardor!

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[2.2] What is nonlinear science?

Stanislaw Ulam reportedly said (something like) "Calling a science 'nonlinear' is like calling zoology 'the study of non-human animals'. So why do we have a name that appears to be merely a negative?

Firstly, linearity is rather special, and no model of a real system is truly linear. Some things are profitably studied as linear approximations to the real models--for example the fact that Hooke's law, the linear law of elasticity (strain is proportional to stress) is approximately valid for a pendulum of small amplitude implies that its period is approximately independent of amplitude. However, as the amplitude gets large the period gets longer, a fundamental effect of nonlinearity in the pendulum equations (see http://monet.physik.unibas.ch/~elmer/pendulum/upend.htm and [3.10]).

(You might protest that quantum mechanics is the fundamental theory and that it is linear! However this is at the expense of infinite dimensionality which is just as bad or worse--and 'any' finite dimensional nonlinear model can be turned into an infinite dimensional linear one--e.g. a map $x' = f(x)$ is equivalent to the linear integral equation often called the Perron-Frobenius equation

$$p'(x) = \text{integral } [\, p(y) \backslash\text{delta}(x-f(y))\, dy \,]$$

Here $p(x)$ is a density, which could be interpreted as the probability of finding oneself at the point x, and the Dirac-delta function effectively moves the points according to the map f to give the new density. So even a nonlinear map is equivalent to a linear operator.)

Secondly, nonlinear systems have been shown to exhibit surprising and complex effects that would never be anticipated by a scientist trained only in linear techniques. Prominent examples of these include bifurcation, chaos, and solitons. Nonlinearity has its most profound effects on dynamical systems (see [2.3]).

Further, while we can enumerate the linear objects, nonlinear ones are nondenumerable, and as of yet mostly unclassified. We currently have no general techniques (and very few special ones) for telling whether a particular nonlinear system will exhibit the complexity of chaos, or the simplicity of order. Thus since we cannot yet subdivide nonlinear science into proper subfields, it exists as a whole.

Nonlinear science has applications to a wide variety of fields, from mathematics, physics, biology, and chemistry, to engineering, economics, and medicine. This is one of its most exciting aspects--that it brings researchers from many disciplines together with a common language.

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[2.3] What is a dynamical system?

A dynamical system consists of an abstract phase space or state space, whose coordinates describe the dynamical state at any instant; and a dynamical rule which specifies the immediate future trend of all state variables, given only the present values of those same state variables. Mathematically, a dynamical system is described by an initial value problem.

Dynamical systems are "deterministic" if there is a unique consequent to every state, and "stochastic" or "random" if there is more than one consequent chosen from some probability distribution (the "perfect" coin toss has two consequents with equal probability for each initial state). Most of nonlinear science--and everything in this FAQ--deals with deterministic systems.

A dynamical system can have discrete or continuous time. The discrete case is defined by a map, $z_1 = f(z_0)$, that gives the state z_1 resulting from the initial state z_0 at the next time value. The continuous case is defined by a "flow", $z(t) = \phi_t(z_0)$, which gives the state at time t, given that the state was z_0 at time 0. A smooth flow can be differentiated w.r.t. time to give a differential equation, $dz/dt = F(z)$. In this case we call $F(z)$ a "vector field," it gives a vector pointing in the direction of the velocity at every point in phase space.

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[2.4] What is phase space?

Phase space is the collection of possible states of a dynamical system. A phase space can be finite (e.g. for the ideal coin toss, we have two states heads and tails), countably infinite (e.g. state variables are integers), or uncountably infinite (e.g. state variables are real numbers). Implicit in the notion is that a particular state in phase space specifies the system completely; it is all we need to know about the system to have complete knowledge of the immediate future. Thus the phase space of the planar pendulum is two-dimensional, consisting of the position (angle) and velocity. According to Newton, specification of these two variables uniquely determines the subsequent motion of the pendulum.

Note that if we have a non-autonomous system, where the map or vector field depends explicitly on time (e.g. a model for plant growth depending on solar flux), then according to our definition of phase space, we must include time as a phase space coordinate--since one must specify a specific time (e.g. 3PM on Tuesday) to know the subsequent motion. Thus $dz/dt = F(z,t)$ is a dynamical system on the phase space consisting of (z,t), with the addition of the new dynamics $dt/dt = 1$.

The path in phase space traced out by a solution of an initial value problem is called an orbit or trajectory of the dynamical system. If the state variables take real values in a continuum, the orbit of a continuous-time system is a curve, while the orbit of a discrete-time system is a sequence of points.

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[2.5] What is a degree of freedom?

The notion of "degrees of freedom" as it is used for Hamiltonian systems means one canonical conjugate pair, a configuration, q, and its conjugate momentum p. Hamiltonian systems (sometimes mist akenly identified with the notion of conservative systems) always have such pairs of variables, and s o the phase space is even dimensional.

In the study of dissipative systems the term "degree of freedom" is often used differently, to mean a single coordinate dimension of the phase space. This can lead to confusion, and it is advisable to ch eck which meaning of the term is intended in a particular context.

Those with a physics background generally prefer to stick with the Hamiltonian definition of the ter m "degree of freedom." For a more general system the proper term is "order" which is equal to the dimension of the phase space.

Note that a dynamical system with N d.o.f. Hamiltonian nominally moves in a $2N$ dimensional pha se space. However, if $H(q,p)$ is time independent, then energy is conserved, and therefore the motio n is really on a $2N$-1 dimensional energy surface, $H(q,p) = E$. Thus e.g. the planar, circular restricte d 3 body problem is 2 d.o.f., and motion is on the 3D energy surface of constant "Jacobi constant." It can be reduced to a 2D area preserving map by Poincaré section (see [2.6]).

If the Hamiltonian is time dependent, then we generally say it has an additional 1/2 degree of freedo m, since this adds one dimension to the phase space. (i.e. 1 1/2 d.o.f. means three variables, q, p and t, and energy is no longer conserved).

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[2.6] What is a map?

A map is simply a function, f, on the phase space that gives the next state, $f(z)$ (the image), of the sys tem given its current state, z. (Often you will find the notation $z' = f(z)$, where the prime means the n ext point, not the derivative.)

Now a function must have a single value for each state, but there could be several different states tha t give rise to the same image. Maps that allow every state in the phase space to be accessed (onto) a nd which have precisely one pre-image for each state (one-to-one) are invertible. If in addition the m ap and its inverse are continuous (with respect to the phase space coordinate z), then it is called a ho meomorphism. A homeomorphism that has at least one continuous derivative (w.r.t. z) and a contin uously differentiable inverse is a diffeomorphism.

Iteration of a map means repeatedly applying the map to the consequents of the previous application . Thus we get a sequence

$$z_n = f(z_{n-1}) = f(f(z_{n-2})...) = f^n(z_0)$$

This sequence is the orbit or trajectory of the dynamical system with initial condition z_0.

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[2.7] How are maps related to flows (differential equations)?

Every differential equation gives rise to a map, the time one map, defined by advancing the flow one unit of time. This map may or may not be useful. If the differential equation contains a term or term s periodic in time, then the time T map (where T is the period) is very useful--it is an example of a P oincaré section. The time T map in a system with periodic terms is also called a stroboscopic map, s ince we are effectively looking at the location in phase space with a stroboscope tuned to the period T. This map is useful because it permits us to dispense with time as a phase space coordinate: the re maining coordinates describe the state completely so long as we agree to consider the same instant within every period.

In autonomous systems (no time-dependent terms in the equations), it may also be possible to defin e a Poincaré section and again reduce the phase space dimension by one. Here the Poincaré section is defined not by a fixed time interval, but by successive times when an orbit crosses a fixed surface in phase space. (Surface here means a manifold of dimension one less than the phase space dimens ion).

However, not every flow has a global Poincaré section (e.g. any flow with an equilibrium point), whi ch would need to be transverse to every possible orbit.

Maps arising from stroboscopic sampling or Poincaré section of a flow are necessarily invertible, b ecause the flow has a unique solution through any point in phase space--the solution is unique both forward and backward in time. However, noninvertible maps can be relevant to differential equation s: Poincaré maps are sometimes very well approximated by noninvertible maps. For example, the H enon map (x,y) -> (-y-a+$x^{2,}bx$) with small |b| is close to the logistic map, x -> -a+x^2.

It is often (though not always) possible to go backwards, from an invertible map to a differential eq uation having the map as its Poincaré map. This is called a suspension of the map. One can also do this procedure approximately for maps that are close to the identity, giving a flow that approximates the map to some order. This is extremely useful in bifurcation theory.

Note that any numerical solution procedure for a differential initial value problem which uses discre te time steps in the approximation is effectively a map. This is not a trivial observation; it helps expl ain for example why a continuous-time system which should not exhibit chaos may have numerical solutions which do--see [2.15].

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[2.8] What is an attractor?

Informally an attractor is simply a state into which a system settles (thus dissipation is needed). Thu s in the long term, a dissipative dynamical system may settle into an attractor.

Interestingly enough, there is still some controversy in the mathematics community as to an appropriate definition of this term. Most people adopt the definition

Attractor: A set in the phase space that has a neighborhood in which every point stays nearby and a pproaches the attractor as time goes to infinity.

Thus imagine a ball rolling inside of a bowl. If we start the ball at a point in the bowl with a velocity too small to reach the edge of the bowl, then eventually the ball will settle down to the bottom of the bowl with zero velocity: thus this equilibrium point is an attractor. The neighborhood of points that eventually approach the attractor is the *basin of attraction* for the attractor. In our example the basin is the set of all configurations corresponding to the ball in the bowl, and for each such point all sma ll enough velocities (it is a set in the four dimensional phase space [2.4]).

Attractors can be simple, as the previous example. Another example of an attractor is a limit cycle, which is a periodic orbit that is attracting (limit cycles can also be repelling). More surprisingl y, attractors can be chaotic (see [2.9]) and/or strange (see [2.12]).

The boundary of a basin of attraction is often a very interesting object since it distinguishes between different types of motion. Typically a basin boundary is a saddle orbit, or such an orbit and its stable manifold. A *crisis* is the change in an attractor when its basin boundary is destroyed.

An alternative definition of attractor is sometimes used because there are systems that have s ets that attract most, but not all, initial conditions in their neighborhood (such phenomena is someti mes called riddling of the basin). Thus, Milnor defines an attractor as a set for which a positive mea sure (probability, if you like) of initial conditions in a neighborhood are asymptotic to the set.

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[2.9] What is chaos?

It has been said that "Chaos is a name for any order that produces confusion in our minds." (George Santayana, thanks to Fred Klingener for finding this). However, the mathematical definition is, roughly speaking,
Chaos: effectively unpredictable long time behavior arising in a deterministic dynamical system because of sensitivity to initial conditions.
It must be emphasized that a deterministic dynamical system is perfectly predictable given perfect knowledge of the initial condition, and is in practice always predictable in the short term. The key to long-term unpredictability is a property known as sensitivity to (or sensitive dependence on) initial conditions.

For a dynamical system to be *chaotic* it must have a 'large' set of initial conditions which are highly unstable. No matter how precisely you measure the initial condition in these systems, your prediction of its subsequent motion goes radically wrong after a short time. Typically (see [2.14] for one definition of 'typical'), the predictability horizon grows only logarithmically with the precision of measurement (for positive Lyapunov exponents, see [2.11]). Thus for each increase in precision by a factor of 10, say, you may only be able to predict two more time units (measured in units of the Lyapunov time, i.e. the inverse of the Lyapunov exponent).

More precisely: A map f is *chaotic* on a compact invariant set S if
(i) f is transitive on S (there is a point x whose orbit is dense in S), and
(ii) f exhibits sensitive dependence on S (see [2.10]).
To these two requirements Devaney adds the requirement that periodic points are dense in S, but this doesn't seem to be really in the spirit of the notion, and is probably better treated as a theorem (very difficult and very important), and not part of the definition.

Usually we would like the set S to be a large set. It is too much to hope for except in special examples that S be the entire phase space. If the dynamical system is dissipative then we hope that S is an attractor (see [2.8]) with a large basin. However, this need not be the case--we can have a chaotic saddle, an orbit that has some unstable directions as well as stable directions.

As a consequence of long-term unpredictability, time series from chaotic systems may appear irregular and disorderly. However, chaos is definitely not (as the name might suggest) complete disorder; it is disorder in a deterministic dynamical system, which is always predictable for short times.

The notion of chaos seems to conflict with that attributed to Laplace: given precise knowledge of the initial conditions, it should be possible to predict the future of the universe. However, Laplace's dictum is certainly true for any deterministic system, recall [2.3]. The main consequence of chaotic motion is that given imperfect knowledge, the predictability horizon in a deterministic system is much shorter than one might expect, due to the exponential growth of errors. The belief that small errors should have small consequences was perhaps engendered by the success of Newton's mechanics applied to planetary motions. Though these happen to be regular on human historic time scales, they are chaotic on the 5 million year time scale (see e.g. "Newton's Clock", by Ivars Peterson (1993 W.H. Freeman).

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[2.10] What is sensitive dependence on initial conditions?

Consider a boulder precariously perched on the top of an ideal hill. The slightest push will cause the boulder to roll down one side of the hill or the other: the subsequent behavior depends sensitively on the direction of the push--and the push can be arbitrarily small. Of course, it is of great importance to you which direction the boulder will go if you are standing at the bottom of the hill on one side or the other!

Sensitive dependence is the equivalent behavior for every initial condition--every point in the phase space is effectively perched on the top of a hill.

More precisely a set S exhibits sensitive dependence if there is an r such that for any $epsilon > 0$ and for each x in S, there is a y such that $|x - y| < epsilon$, and $|x\_n - y\_n| > r$ for some $n > 0$. Then there is a fixed distance r (say 1), such that no matter how precisely one specifies an initial state there are nearby states that eventually get a distance r away.

Note: sensitive dependence does not require exponential growth of perturbations (positive Lyapunov exponent), but this is typical (see [2.14]) for chaotic systems. Note also that we most definitely do not require ALL nearby initial points diverge--generically [2.14] this does not happen--some nearby points may converge. (We may modify our hilltop analogy slightly and say that every point in phase space acts like a high mountain pass.) Finally, the words "initial conditions" are a bit misleading: a typical small disturbance introduced at any time will grow similarly. Think of "initial" as meaning "a time when a disturbance or error is introduced," not necessarily time zero.

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[2.11] What are Lyapunov exponents?

(Thanks to Ronnie Mainieri & Fred Klingener for contributing to this answer)

The hardest thing to get right about Lyapunov exponents is the spelling of Lyapunov, which you will variously find as Liapunov, Lyapunof and even Liapunoff. Of course Lyapunov is really spelled in the Cyrillic alphabet: (Lambda)(backwards r)(pi)(Y)(H)(0)(B). Now that there is an ANSI standard of transliteration for Cyrillic, we expect all references to converge on the version Lyapunov.

Lyapunov was born in Russia in 6 June 1857. He was greatly influenced by Chebyshev and was a student with Markov. He was also a passionate man: Lyapunov shot himself the day his wife died. He died 3 Nov. 1918, three days later. According to the request on a note he left, Lyapunov was buried with his wife. [biographical data from a biography by A. T. Grigorian].

Lyapunov left us with more than just a simple note. He left a collection of papers on the equilibrium shape of rotating liquids, on probability, and on the stability of low-dimensional dynamical systems. It was from his dissertation that the notion of Lyapunov exponent emerged. Lyapunov was interested in showing how to discover if a solution to a dynamical system is stable or not for all times. The usual method of studying stability, i.e. linear stability, was not good enough, because if you waited long enough the small errors due to linearization would pile up and make the approximation invalid. Lyapunov developed concepts (now called Lyapunov Stability) to overcome these difficulties.

Lyapunov exponents measure the rate at which nearby orbits converge or diverge. There are as many Lyapunov exponents as there are dimensions in the state space of the system, but the largest is usually the most important. Roughly speaking the (maximal) Lyapunov exponent is the time constant, lambda, in the expression for the distance between two nearby orbits, exp($lambda * t$). If lambda is negative, then the orbits converge in time, and the dynamical system is insensitive to initial conditions. However, if lambda is positive, then the distance between nearby orbits grows exponentially in time, and the system exhibits sensitive dependence on initial conditions.

There are basically two ways to compute Lyapunov exponents. In one way one chooses two nearby points, evolves them in time, measuring the growth rate of the distance between them. This is useful when one has a time series, but has the disadvantage that the growth rate is really not a local effect a s the points separate. A better way is to measure the growth rate of tangent vectors to a given orbit.

More precisely, consider a map *f* in an *m* dimensional phase space, and its derivative matrix *Df*(*x*). L et *v* be a tangent vector at the point *x*. Then we define a function

```
                                1          n
        L(x,v)  =     lim      --- ln |( Df (x)v )|
                   n -> oo      n
```

Now the Multiplicative Ergodic Theorem of Oseledec states that this limit exists for almost all point s x and all tangent vectors *v*. There are at most m distinct values of *L* as we let *v* range over the tange nt space. These are the Lyapunov exponents at *x*.

For more information on computing the exponents see

Wolf, A., J. B. Swift, et al. (1985). "Determining Lyapunov Exponents from a Time Series." Phys ica D **16**: 285-317.
Eckmann, J.-P., S. O. Kamphorst, et al. (1986). "Liapunov exponents from time series." Phys. Re v. A **34**: 4971-4979.

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[2.12] What is a Strange Attractor?

Before Chaos (BC?), the only known attractors (see [2.8]) were fixed points, periodic orbits (limit cycles), and invariant tori (quasiperiodic orbits). In fact the famous Poincaré-Bendixson theo rem states that for a pair of first order differential equations, only fixed points and limit cycles can occur (there is no chaos in 2D flows).

In a famous paper in 1963, Ed Lorenz discovered that simple systems of three differential e quations can have complicated attractors. The Lorenz attractor (with its butterfly wings reminding u s of sensitive dependence (see [2.10])) is the "icon" of chaos http://kong.apmaths.uwo.ca/~bfraser/v ersion1/lorenzintro.html. Lorenz showed that his attractor was chaotic, since it exhibited sensitive de pendence. Moreover, his attractor is also "strange," which means that it is a fractal (see [3.2]).

The term strange attractor was introduced by Ruelle and Takens in 1970 in their discussion of a scenario for the onset of turbulence in fluid flow. They noted that when periodic motion goes u nstable (with three or more modes), the typical (see [2.14]) result will be a geometrically strange obj ect.

Unfortunately, the term strange attractor is often used for any chaotic attractor. However, the term should be reserved for attractors that are "geometrically" strange, e.g. fractal. One can have ch aotic attractors that are not strange (a trivial example would be to take a system like the cat map, whi ch has the whole plane as a chaotic set, and add a third dimension which is simply contracting onto t he plane). There are also strange, nonchaotic attractors (see Grebogi, C., et al. (1984). "Strange Attr actors that are not Chaotic." Physica D **13**: 261-268).

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[2.13] Can computers simulate chaos?

Strictly speaking, chaos cannot occur on computers because they deal with finite sets of numbers. T hus the initial condition is always precisely known, and computer experiments are perfectly predicta ble, in principle. In particular because of the finite size, every trajectory computed will eventually hav e to repeat (an thus be eventually periodic). On the other hand, computers can effectively simulate c haotic behavior for quite long times (just so long as the discreteness is not noticeable). In particular if one uses floating point numbers in double precision to iterate a map on the unit square, then there are about 10^28 different points in the phase space, and one would expect the "typical" chaotic orbit to have a period of about 10^14 (this square root of the number of points estimate is given by Rannou for random diffeomorphisms and does not really apply to floating point operations, but nonetheless the period should be a big number). See, e.g.,

Earn, D. J. D. and S. Tremaine, "Exact Numerical Studies of Hamiltonian Maps: Iterating without Roundoff Error," Physica D **56**, 1-22 (1992).

Binder, P. M. and R. V. Jensen, "Simulating Chaotic Behavior with Finite State Machines," Phys. Rev. **34A**, 4460-3 (1986).

Rannou, F., "Numerical Study of Discrete Plane Area-Preserving Mappings," Astron. and Astrophys. **31**, 289-301 (1974).

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[2.14] What is generic?

(Thanks to Hawley Rising for contributing to this answer)

Generic in dynamical systems is intended to convey "usual" or, more properly, "observable". Roughly speaking, a property is generic over a class if any system in the class can be modified ever so slightly (perturbed), into one with that property.

The formal definition is done in the language of topology: Consider the class to be a space of systems, and suppose it has a *topology* (some notion of a neighborhood, or an open set). A subset of this space is *dense* if its *closure* (the subset plus the limits of all sequences in the subset) is the whole space. It is *open* and *dense* if it is also an open set (union of neighborhoods). A set is *countable* if it can be put into 1-1 correspondence with the counting numbers. A *countable intersection of open dense sets* is the intersection of a countable number of open dense sets. If all such intersections in a space are also dense, then the space is called a *Baire* space, which basically means it is big enough. If we have such a Baire space of dynamical systems, and there is a property which is true on a countable intersection of open dense sets, then that property is *generic*.

If all this sounds too complicated, think of it as a precise way of defining a set which is near every system in the collection (dense), which isn't too big (need not have any "regions" where the property is true for *every* system). Generic is much weaker than "almost everywhere" (occurs with probability 1), in fact, it is possible to have generic properties which occur with probability zero. But it is as strong a property as one can define topologically, without having to have a property hold true in a region, or talking about measure (probability), which isn't a topological property (a property preserved by a continuous function).

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[2.15] What is the minimum phase space dimension for chaos?

This is a slightly confusing topic, since the answer depends on the type of system considered. First consider a flow (or system of differential equations). In this case the Poincaré-Bendixson theorem tells us that there is no chaos in one or two-dimensional phase spaces. Chaos is possible in three-dimensional flows--standard examples such as the Lorenz equations are indeed three-dimensional, and there are mathematical 3D flows that are provably chaotic (e.g. the 'solenoid').

Note: if the flow is non-autonomous then time is a phase space coordinate, so a system with two physical variables + time becomes three-dimensional, and chaos is possible (i.e. Forced second-order oscillators do exhibit chaos.)

For maps, it is possible to have chaos in one dimension, but only if the map is not invertible. A prominent example is the Logistic map

```
x' = f(x) = rx(1-x).
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This is provably chaotic for $r = 4$, and many other values of r as well (see e.g. Devaney). Note that every point $x < f(1/2)$ has two preimages, so this map is not invertible.

For homeomorphisms, we must have at least two-dimensional phase space for chaos. This is equivalent to the flow result, since a three-dimensional flow gives rise to a two-dimensional homeomorphism by Poincaré section (see [2.7]).

Note that a numerical algorithm for a differential equation is a map, because time on the computer is necessarily discrete. Thus numerical solutions of two and even one dimensional systems of ordinary differential equations may exhibit chaos. Usually this results from choosing the size of the time step too large. For example Euler discretization of the Logistic differential equation, $dx/dt = rx(1-x)$, is equivalent to the logistic map. See e.g. S. Ushiki, "Central difference scheme and chaos," Physica **4 D** (1982) 407-424.

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[3] Applications and Advanced Theory

[3.1] What are complex systems?

(Thanks to Troy Shinbrot for contributing to this answer)

Complex systems are spatially and/or temporally extended nonlinear systems characterized by collective properties associated with the system as a whole--and that are different from the characteristic behaviors of the constituent parts.

While, chaos is the study of how simple systems can generate complicated behavior, complexity is the study of how complicated systems can generate simple behavior. An example of complexity is the synchronization of biological systems ranging from fireflies to neurons (e.g. Matthews, PC, Mirollo, RE & Strogatz, SH "Dynamics of a large system of coupled nonlinear oscillators," Physica **52D** (1991) 293-331). In these problems, many individual systems conspire to produce a single collective rhythm.

The notion of complex systems has received lots of popular press, but it is not really clear as of yet if there is a "theory" about a "concept". We are withholding judgment. See

http://www.calresco.org/index.htm The Complexity & Artificial Life Web Site
http://www.calresco.org/sos/sosfaq.htm The self-organized systems FAQ

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[3.2] What are fractals?

One way to define "fractal" is as a negation: a fractal is a set that does not look like a Euclidean object (point, line, plane, etc.) no matter how closely you look at it. Imagine focusing in on a smooth curve (imagine a piece of string in space)--if you look at any piece of it closely enough it eventually looks like a straight line (ignoring the fact that for a real piece of string it will soon look like a cylinder and eventually you will see the fibers, then the atoms, etc.). A fractal, like the Koch Snowflake, which is topologically one dimensional, never looks like a straight line, no matter how closely you look. There are indentations, like bays in a coastline; look closer and the bays have inlets, closer still the inlets have subinlets, and so on. Simple examples of fractals include Cantor sets (see [3.5], Sierpinski curves, the Mandelbrot set and (almost surely) the Lorenz attractor (see [2.12]). Fractals also approximately describe many real-world objects, such as clouds (see http://makeashorterlink.com/?Z50D42C16) mountains, turbulence, coastlines, roots and branches of trees and veins and lungs of animals.

"Fractal" is a term which has undergone refinement of definition by a lot of people, but was first coined by B. Mandelbrot, http://physics.hallym.ac.kr/reference/physicist/Mandelbrot.html, and defined as a set with fractional (non-integer) dimension (Hausdorff dimension, see [3.4]). Mandelbrot defines a fractal in the following way:

> A geometric figure or natural object is said to be fractal if it combines the following characteristics: (a) its parts have the same form or structure as the whole, except that they are at a different scale and may be slightly deformed; (b) its form is extremely irregular, or extremely interrupted or fragmented, and remains so, whatever the scale of examination; (c) it contains "distinct elements" whose scales are very varied and cover a large range." (Les Objets Fractales 1989, p.154)

See the extensive FAQ from sci.fractals at
<ftp://rtfm.mit.edu/pub/usenet/news.answers/fractal-faq

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[3.3] What do fractals have to do with chaos?

Often chaotic dynamical systems exhibit fractal structures in phase space. However, there is no direct relation. There are chaotic systems that have nonfractal limit sets (e.g. Arnold's cat map) and fractal structures that can arise in nonchaotic dynamics (see e.g. Grebogi, C., et al. (1984). "Strange Attractors that are not Chaotic." Physica 13**D**: 261-268.)

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[3.4] What are topological and fractal dimension?

See the fractal FAQ:
ftp://rtfm.mit.edu/pub/usenet/news.answers/fractal-faq
or the site
http://pro.wanadoo.fr/quatuor/mathematics.htm

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[3.5] What is a Cantor set?

(Thanks to Pavel Pokorny for contributing to this answer)

A Cantor set is a surprising set of points that is both infinite (uncountably so, see [2.14]) and yet diffuse. It is a simple example of a fractal, and occurs, for example as the strange repellor in the logistic map (see [2.15]) when r>4. The standard example of a Cantor set is the "middle thirds" set constructed on the interval between 0 and 1. First, remove the middle third. Two intervals remain, each one of length one third. From each remaining interval remove the middle third. Repeat the last step infinitely many times. What remains is a Cantor set.

More generally (and abstrusely) a Cantor set is defined topologically as a nonempty, compact set which is perfect (every point is a limit point) and totally disconnected (every pair of points in the set are contained in disjoint covering neighborhoods).

See also
http://www.shu.edu/html/teaching/math/reals/topo/defs/cantor.html
http://personal.bgsu.edu/~carother/cantor/Cantor1.html
http://mizar.uwb.edu.pl/JFM/Vol7/cantor_1.html

Georg Ferdinand Ludwig Philipp Cantor was born 3 March 1845 in St Petersburg, Russia, and died 6 Jan 1918 in Halle, Germany. To learn more about him see:
http://turnbull.dcs.st-and.ac.uk/history/Mathematicians/Cantor.html
http://www.shu.edu/html/teaching/math/reals/history/cantor.html

To read more about the Cantor function (a function that is continuous, differentiable, increasing, non-constant, with a derivative that is zero everywhere except on a set with length zero) see
http://www.shu.edu/projects/reals/cont/fp_cantr.html

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[3.6] What is quantum chaos?

(Thanks to Leon Poon for contributing to this answer)

According to the correspondence principle, there is a limit where classical behavior as described by Hamilton's equations becomes similar, in some suitable sense, to quantum behavior as described by the appropriate wave equation. Formally, one can take this limit to be h -> 0, where h is Planck's con stant; alternatively, one can look at successively higher energy levels. Such limits are referred to as " semiclassical". It has been found that the semiclassical limit can be highly nontrivial when the classi cal problem is chaotic. The study of how quantum systems, whose classical counterparts are chaotic , behave in the semiclassical limit has been called quantum chaos. More generally, these considerati ons also apply to elliptic partial differential equations that are physically unrelated to quantum consi derations. For example, the same questions arise in relating classical waves to their corresponding r ay equations. Among recent results in quantum chaos is a prediction relating the chaos in the classic al problem to the statistics of energy-level spacings in the semiclassical quantum regime.

Classical chaos can be used to analyze such ostensibly quantum systems as the hydrogen atom, wh ere classical predictions of microwave ionization thresholds agree with experiments. See Koch, P. M. and K. A. H. van Leeuwen (1995). "Importance of Resonances in Microwave Ionization of Exci ted Hydrogen Atoms." Physics Reports **255**: 289-403.

See also:
http://sagar.physics.neu.edu/qchaos/qc.html Quantum Chaos
http://www.mpipks-dresden.mpg.de/~noeckel/microlasers.html Microlaser Cavities

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[3.7] How do I know if my data are deterministic?

(Thanks to Justin Lipton for contributing to this answer)

How can I tell if my data is deterministic? This is a very tricky problem. It is difficult because in pra ctice no time series consists of pure 'signal.' There will always be some form of corrupting noise, ev en if it is present as round-off or truncation error or as a result of finite arithmetic or quantization. T hus any real time series, even if mostly deterministic, will be a stochastic processes

All methods for distinguishing deterministic and stochastic processes rely on the fact that a determi nistic system will always evolve in the same way from a given starting point. Thus given a time serie s that we are testing for determinism we
(1) pick a test state
(2) search the time series for a similar or 'nearby' state and
(3) compare their respective time evolution.

Define the error as the difference between the time evolution of the 'test' state and the time evolution of the nearby state. A deterministic system will have an error that either remains small (stable, regula r solution) or increase exponentially with time (chaotic solution). A stochastic system will have a ra ndomly distributed error.

Essentially all measures of determinism taken from time series rely upon finding the closest states t o a given 'test' state (i.e., correlation dimension, Lyapunov exponents, etc.). To define the state of a s ystem one typically relies on phase space embedding methods, see [3.14].

Typically one chooses an embedding dimension, and investigates the propagation of the error betwe en two nearby states. If the error looks random, one increases the dimension. If you can increase th e dimension to obtain a deterministic looking error, then you are done. Though it may sound simple it is not really! One complication is that as the dimension increases the search for a nearby state req uires a lot more computation time and a lot of data (the amount of data required increases exponenti ally with embedding dimension) to find a suitably close candidate. If the embedding dimension (nu mber of measures per state) is chosen too small (less than the 'true' value) deterministic data can appear to be random but in theory there is no problem choosing the dimension too large--the method will work. Practically, anything approaching about 10 dimensions is considered so large that a stochastic description is probably more suitable and convenient anyway.

See e.g.,
Sugihara, G. and R. M. May (1990). "Nonlinear Forecasting as a Way of Distinguishing Chaos from Measurement Error in Time Series." Nature **344**: 734-740.

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[3.8] What is the control of chaos?

Control of chaos has come to mean the two things:
- stabilization of unstable periodic orbits,
- use of recurrence to allow stabilization to be applied locally.

Thus term "control of chaos" is somewhat of a misnomer--but the name has stuck. The ideas for controlling chaos originated in the work of Hubler followed by the Maryland Group.

Hubler, A. W. (1989). "Adaptive Control of Chaotic Systems." Helv. Phys. Acta **62**: 343-346.
Ott, E., C. Grebogi, et al. (1990). "Controlling Chaos." Physical Review Letters **64**(11): 1196-1199. http://www-chaos.umd.edu/publications/abstracts.html#prl64.1196

The idea that chaotic systems can in fact be controlled may be counterintuitive--after all they are unpredictable in the long term. Nevertheless, numerous theorists have independently developed methods which can be applied to chaotic systems, and many experimentalists have demonstrated that physical chaotic systems respond well to both simple and sophisticated control strategies. Applications have been proposed in such diverse areas of research as communications, electronics, physiology, epidemiology, fluid mechanics and chemistry.

The great bulk of this work has been restricted to low-dimensional systems; more recently, a few researchers have proposed control techniques for application to high- or infinite-dimensional systems. The literature on the subject of the control of chaos is quite voluminous; nevertheless several reviews of the literature are available, including:

Shinbrot, T. Ott, E., Grebogi, C. & Yorke, J.A., "Using Small Perturbations to Control Chaos," Nature, **363** (1993) 411-7.
Shinbrot, T., "Chaos: Unpredictable yet Controllable?" Nonlin. Sciences Today, **3:2** (1993) 1-8.
Shinbrot, T., "Progress in the Control of Chaos," Advance in Physics (in press).
Ditto, WL & Pecora, LM "Mastering Chaos," Scientific American (**Aug. 1993**), 78-84.
Chen, G. & Dong, X, "From Chaos to Order -- Perspectives and Methodologies in Controlling Chaotic Nonlinear Dynamical Systems," Int. J. Bif. & Chaos **3** (1993) 1363-1409.

It is generically quite difficult to control high dimensional systems; an alternative approach is to use control to reduce the dimension before applying one of the above techniques. This approach is in its infancy; see:

Auerbach, D., Ott, E., Grebogi, C., and Yorke, J.A. "Controlling Chaos in High Dimensional Systems," Phys. Rev. Lett. **69** (1992) 3479-82
http://www-chaos.umd.edu/publications/abstracts.html#prl69.3479

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[3.9] How can I build a chaotic circuit?

(Thanks to Justin Lipton and Jose Korneluk for contributing to this answer)

There are many different physical systems which display chaos, dripping faucets, water wheels, osci llating magnetic ribbons etc. but the most simple systems which can be easily implemented are chao tic circuits. In fact an electronic circuit was one of the first demonstrations of chaos which showed t hat chaos is not just a mathematical abstraction. Leon Chua designed the circuit 1983.

The circuit he designed, now known as Chua's circuit, consists of a piecewise linear resistor as its n onlinearity (making analysis very easy) plus two capacitors, one resistor and one inductor--the circu it is unforced (autonomous). In fact the chaotic aspects (bifurcation values, Lyapunov exponents, va rious dimensions etc.) of this circuit have been extensively studied in the literature both experimenta lly and theoretically. It is extremely easy to build and presents beautiful attractors (see [2.8]) (the m ost famous known as the double scroll attractor) that can be displayed on a CRO.

For more information on building such a circuit try: see

http://www.cmp.caltech.edu/~mcc/chaos_new/Chua.html Chua's Circuit Applet

References

Matsumoto T. and Chua L.O. and Komuro M. "Birth and Death of the Double Scroll" Physica **D24** 97-124, 1987.

Kennedy M. P., "Robust OP Amp Realization of Chua's Circuit", Frequenz **46**, no. 3-4, 1992

Madan, R. A., Chua's Circuit: A paradigm for chaos, ed. R. A. Madan, Singapore: World Scientific, 1993.

Pecora, L. and Carroll, T. Nonlinear Dynamics in Circuits, Singapore: World Scientific, 1995.

Nonlinear Dynamics of Electronic Systems, Proceedings of the Workshop NDES 1993, A.C.Davies and W.Schwartz, eds., World Scientific, 1994, ISBN 981-02-1769-2.

Parker, T.S., and L.O.Chua, Practical Numerical Algorithms for Chaotic Systems, Springer-Verlag, 1989, ISBN's: 0-387-96689-7 and 3-540-96689-7.

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[3.10] What are simple experiments to demonstrate chaos?

There are many "chaos toys" on the market. Most consist of some sort of pendulum that is forced by an electromagnet. One can of course build a simple double pendulum to observe beautiful chaoti c behavior see

http://quasar.mathstat.uottawa.ca/~selinger/lagrange/doublependulum.html Experimental Pendulu m Designs

http://www.maths.tcd.ie/~plynch/SwingingSpring/doublependulum.html Java Applet

http://monet.physik.unibas.ch/~elmer/pendulum/ Java Applets Pendulum Lab

My favorite double pendulum consists of two identical planar pendula, so that you can demonstrate sensitive dependence [2.10], for a Java applet simulation see http://www.cs.mu.oz.au/~mkwan/pend ulum/pendulum.html. Another cute toy is the "Space Circle" that you can find in many airport gift s hops. This is discussed in the article:

A. Wolf & T. Bessoir, Diagnosing Chaos in the Space Circle, Physica **50D**, 1991.

One of the simplest chemical systems that shows chaos is the Belousov-Zhabotinsky reaction. The book by Strogatz [4.1] has a good introduction to this subject,. For the recipe see http://www.ux.his.no/~ruoff/BZ_Phenomenology.html. Chemical chaos is modeled (in a generic sense) by the "Brusselator" system of differential equations. See

Nicolis, Gregoire & Prigogine, (1989) Exploring Complexity: An Introduction W. H. Freeman

The Chaotic waterwheel, while not so simple to build, is an exact realization of Lorenz famous equations. This is nicely discussed in Strogatz book [4.1] as well.

Billiard tables can exhibit chaotic motion, see http://www.maa.org/mathland/mathland_3_3.html, though it might be hard to see this next time you are in a bar, since a rectangular table is not chaotic!

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[3.11] What is targeting?

(Thanks to Serdar Iplikçi for contributing to this answer)

Targeting is the task of steering a chaotic system from any initial point to the target, which can be either an unstable equilibrium point or an unstable periodic orbit, in the shortest possible time, by applying relatively small perturbations. In order to effectively control chaos, [3.8] a targeting strategy is important. See:

Kostelich, E., C. Grebogi, E. Ott, and J. A. Yorke, "Higher Dimensional Targeting," Phys Rev. E,. **47**, , 305-310 (1993).
Barreto, E., E. Kostelich, C. Grebogi, E. Ott, and J. A. Yorke, "Efficient Switching Between Controlled Unstable Periodic Orbits in Higher Dimensional Chaotic Systems," Phys Rev E, **51**, 4169-4172 (1995).

One application of targeting is to control a spacecraft's trajectory so that one can find low energy orbits from one planet to another. Recently targeting techniques have been used in the design of trajectories to asteroids and even of a grand tour of the planets. For example,

E. Bollt and J. D. Meiss, "Targeting Chaotic Orbits to the Moon Through Recurrence," Phys. Lett. A **204**, 373-378 (1995).
http://www.cds.caltech.edu/~marsden/software/spacecraft_orbits.html

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[3.12] What is time series analysis?

(Thanks to Jim Crutchfield for contributing to this answer)

This is the application of dynamical systems techniques to a data series, usually obtained by "measuring" the value of a single observable as a function of time. The major tool in a dynamicist's toolkit is "delay coordinate embedding" which creates a phase space portrait from a single data series. It seems remarkable at first, but one can reconstruct a picture equivalent (topologically) to the full Lorenz attractor (see [2.12])in three-dimensional space by measuring only one of its coordinates, say x(t), and plotting the delay coordinates ($x(t)$, $x(t+h)$, $x(t+2h)$) for a fixed h.

It is important to emphasize that the idea of using derivatives or delay coordinates in time series modeling is nothing new. It goes back at least to the work of Yule, who in 1927 used an autoregressive (AR) model to make a predictive model for the sunspot cycle. AR models are nothing more than delay coordinates used with a linear model. Delays, derivatives, principal components, and a variety of other methods of reconstruction have been widely used in time series analysis since the early 50's, a nd are described in several hundred books. The new aspects raised by dynamical systems theory ar e (i) the implied geometric view of temporal behavior and (ii) the existence of "geometric invariants" , such as dimension and Lyapunov exponents. The central question was not whether delay coordinat es are useful for time series analysis, but rather whether reconstruction methods preserve the geome try and the geometric invariants of dynamical systems. (Packard, Crutchfield, Farmer & Shaw)

G.U. Yule, Phil. Trans. R. Soc. London A **226** (1927) p. 267.
N.H. Packard, J.P. Crutchfield, J.D. Farmer, and R.S. Shaw, "Geometry from a time series", Phys. Rev. Lett. **45**, no. 9 (1980) 712.
F. Takens, "Detecting strange attractors in fluid turbulence", in: Dynamical Systems and Turbulence, eds. D. Rand and L.-S. Young (Springer, Berlin, 1981)
Abarbanel, H.D.I., Brown, R., Sidorowich, J.J., and Tsimring, L.Sh.T. "The analysis of observed chaotic data in physical systems", Rev. Modern Physics **65** (1993) 1331-1392.
D. Kaplan and L. Glass (1995). Understanding Nonlinear Dynamics, Springer-Verlag http://www.cnd.mcgill.ca/books_understanding.html
E. Peters (1994) Fractal Market Analysis : Applying Chaos Theory to Investment and Economics, Wiley http://www.wiley.com/WileyCDA/WileyTitle/productCd-0471585246.html

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[3.13] Is there chaos in the stock market?

(Thanks to Bruce Stewart for Contributions to this answer)

In order to address this question, we must first agree what we mean by chaos, see [2.9].

In dynamical systems theory, chaos means irregular fluctuations in a deterministic system (see [2.3] and [3.7]). This means the system behaves irregularly because of its own internal logic, not becaus e of random forces acting from outside. Of course, if you define your dynamical system to be the s ocio-economic behavior of the entire planet, nothing acts randomly from outside (except perhaps th e occasional meteor), so you have a dynamical system. But its dimension (number of state variables --see [2.4]) is vast, and there is no hope of exploiting the determinism. This is high-dimensional cha os, which might just as well be truly random behavior. In this sense, the stock market is chaotic, but who cares?

To be useful, economic chaos would have to involve some kind of collective behavior which can be f ully described by a small number of variables. In the lingo, the system would have to be self-organi zing, resulting in low- dimensional chaos. If this turns out to be true, then you can exploit the low- d imensional chaos to make short-term predictions. The problem is to identify the state variables whic h characterize the collective modes. Furthermore, having limited the number of state variables, many events now become external to the system, that is, the system is operating in a changing environmen t, which makes the problem of system identification very difficult.

If there were such collective modes of fluctuation, market players would probably know about them; economic theory says that if many people recognized these patterns, the actions they would take to exploit them would quickly nullify the patterns. Market participants would probably not need to kn ow chaos theory for this to happen. Therefore if these patterns exist, they must be hard to recognize because they do not emerge clearly from the sea of noise caused by individual actions; or the patter ns last only a very short time following some upset to the markets; or both.

A number of people and groups have tried to find these patterns. So far the published results are ne gative. There are also commercial ventures involving prominent researchers in the field of chaos; we have no idea how well they are succeeding, or indeed whether they are looking for low-dimensional chaos. In fact it seems unlikely that markets remain stationary long enough to identify a chaotic attr actor (see [2.12]). If you know chaos theory and would like to devote yourself to the rhythms of ma rket trading, you might find a trading firm which will give you a chance to try your ideas. But don't expect them to give you a share of any profits you may make for them :-) !

In short, anyone who tells you about the secrets of chaos in the stock market doesn't know anything useful, and anyone who knows will not tell. It's an interesting question, but you're unlikely to find t he answer.

On the other hand, one might ask a more general question: is market behavior adequately described by linear models, or are there signs of nonlinearity in financial market data? Here the prospect is mo re favorable. Time series analysis (see [3.14]) has been applied these tests to financial data; the resu lts often indicate that nonlinear structure is present. See e.g. the book by Brock, Hsieh, LeBaron, "N onlinear Dynamics, Chaos, and Instability", MIT Press, 1991; and an update by B. LeBaron, "Chao s and nonlinear forecastability in economics and finance," Philosophical Transactions of the Royal Society, Series A, vol 348, Sept 1994, pp 397-404. This approach does not provide a formula for m aking money, but it is stimulating some rethinking of economic modeling. A book by Richard M. G oodwin, "Chaotic Economic Dynamics," Oxford UP, 1990, begins to explore the implications for b usiness cycles.

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[3.14] What are solitons?

The process of obtaining a solution of a linear (constant coefficient) differential equations is simplif ied by the Fourier transform (it converts such an equation to an algebraic equation, and we all know that algebra is easier than calculus!); is there a counterpart which similarly simplifies nonlinear equa tions? The answer is No. Nonlinear equations are qualitatively more complex than linear equations, and a procedure which gives the dynamics as simply as for linear equations must contain a mistake. There are, however, exceptions to any rule.

Certain nonlinear differential equations can be fully solved by, e.g., the "inverse scattering method." Examples are the Korteweg-de Vries, nonlinear Schrodinger, and sine-Gordon equations. In these c ases the real space maps, in a rather abstract way, to an inverse space, which is comprised of continu ous and discrete parts and evolves linearly in time. The continuous part typically corresponds to rad iation and the discrete parts to stable solitary waves, i.e. pulses, which are called solitons. The linear evolution of the inverse space means that solitons will emerge virtually unaffected from interactions with anything, giving them great stability.

More broadly, there is a wide variety of systems which support stable solitary waves through a bala nce of dispersion and nonlinearity. Though these systems may not be integrable as above, in many cases they are close to systems which are, and the solitary waves may share many of the stability pr operties of true solitons, especially that of surviving interactions with other solitary waves (mostly) unscathed. It is widely accepted to call these solitary waves solitons, albeit with qualifications.

Why solitons? Solitons are simply a fundamental nonlinear wave phenomenon. Many very basic lin ear systems with the addition of the simplest possible or first order nonlinearity support solitons; th is universality means that solitons will arise in many important physical situations. Optical fibers ca n support solitons, which because of their great stability are an ideal medium for transmitting infor mation. In a few years long distance telephone communications will likely be carried via solitons.

The soliton literature is by now vast. Two books which contain clear discussions of solitons as well as references to original papers are

A. C. Newell, Solitons in Mathematics and Physics, SIAM, Philadelphia, Penn. (1985)

M.J. Ablowitz and P.A.Clarkson, Solitons, nonlinear evolution equations and inverse scattering, Cambridge (1991). http://www.cup.org/titles/catalogue.asp?isbn=0521387302

See http://www.ma.hw.ac.uk/solitons/

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[3.15] What is spatio-temporal chaos?

Spatio-temporal chaos occurs when system of coupled dynamical systems gives rise to dyn amical behavior that exhibits both spatial disorder (as in rapid decay of spatial correlations) and tem poral disorder (as in nonzero Lyapunov exponents). This is an extremely active, and rather unsettled area of research. For an introduction see:

Cross, M. C. and P. C. Hohenberg (1993). "Pattern Formation outside of Equilibrium." Rev. Mod. Phys. **65**: 851-1112.

http://www.cmp.caltech.edu/~mcc/st_chaos.html Spatio-Temporal Chaos

An interesting application which exhibits pattern formation and spatio-temporal chaos is to excitable media in biological or chemical systems. See

Chaos, Solitions and Fractals **5** #3&4 (1995) Nonlinear Phenomena in Excitable Physiological System, http://www.elsevier.nl/locate/chaos

http://ojps.aip.org/journal_cgi/dbt?KEY=CHAOEH&Volume=8&Issue=1 Chaos focus issue on Fibrillation

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[3.16] What are cellular automata?

(Thanks to Pavel Pokorny for Contributions to this answer)

A Cellular automaton (CA) is a dynamical system with discrete time (like a map, see [2.6]), discrete state space and discrete geometrical space (like an ODE), see [2.7]). Thus they can be repre sented by a state $s(i,j)$ for spatial state i, at time j, where s is taken from some finite set. The update r ule is that the new state is some function of the old states, $s(i,j+1) = f(s)$. The following table shows the distinctions between PDE's, ODE's, coupled map lattices (CML) and CA in taking time, state sp ace or geometrical space either continuous (C) of discrete (D):

	time	state space	geometrical space
PDE	C	C	C
ODE	C	C	D
CML	D	C	D
CA	D	D	D

Perhaps the most famous CA is Conway's game "life." This CA evolves according to a deter ministic rule which gives the state of a site in the next generation as a function of the states of neigh boring sites in the present generation. This rule is applied to all sites.

For further reading see

S. Wolfram (1986) Theory and Application of Cellular Automata, World Scientific Singapore.
Physica **10D** (1984)--the entire volume

Some programs that do CA, as well as more generally "artificial life" are available at
http://www.alife.org/links.html
http://www.kasprzyk.demon.co.uk/www/ALHome.html

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[3.17] What is a Bifurcation?

(Thanks to Zhen Mei for Contributions to this answer)

A bifurcation is a qualitative change in dynamics upon a small variation in the parameters of a system.

Many dynamical systems depend on parameters, e.g. Reynolds number, catalyst density, temperature, etc. Normally a gradually variation of a parameter in the system corresponds to the gradual variation of the solutions of the problem. However, there exists a large number of problems for which the number of solutions changes abruptly and the structure of solution manifolds varies dramatically when a parameter passes through some critical values. For example, the abrupt buckling of a stab when the stress is increased beyond a critical value, the onset of convection and turbulence when the flow parameters are changed, the formation of patterns in certain PDE's, etc. This kind of phenomena is called bifurcation, i.e. a qualitative change in the behavior of solutions of a dynamics system, a partial differential equation or a delay differential equation.

Bifurcation theory is a method for studying how solutions of a nonlinear problem and their stability change as the parameters varies. The onset of chaos is often studied by bifurcation theory. For example, in certain parameterized families of one dimensional maps, chaos occurs by infinitely many period doubling bifurcations

(See http://www.stud.ntnu.no/~berland/math/feigenbaum/)

There are a number of well constructed computer tools for studying bifurcations. In particular see [5.2] for AUTO and DStool.

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[3.18] What is a Hamiltonian Chaos?

The transition to chaos for a Hamiltonian (conservative) system is somewhat different than that for a dissipative system (recall [2.5]). In an integrable (nonchaotic) Hamiltonian system, the motion is "quasiperiodic", that is motion that is oscillatory, but involves more than one independent frequency (see also [2.12]). Geometrically the orbits move on tori, i.e. the mathematical generalization of a donut. Examples of integrable Hamiltonian systems include harmonic oscillators (simple mass on a spring, or systems of coupled linear springs), the pendulum, certain special tops (for example the Euler and Lagrange tops), and the Kepler motion of one planet around the sun.

It was expected that a typical perturbation of an integrable Hamiltonian system would lead to "ergodic" motion, a weak version of chaos in which all of phase space is covered, but the Lyapunov exponents [2.11] are not necessarily positive. That this was not true was rather surprisingly discovered by one of the first computer experiments in dynamics, that of Fermi, Pasta and Ulam. They showed that trajectories in nonintegrable system may also be surprisingly stable. Mathematically this was shown to be the case by the celebrated theorem of Kolmogorov Arnold and Moser (KAM), first proposed by Kolmogorov in 1954. The KAM theorem is rather technical, but in essence says that many of the quasiperiodic motions are preserved under perturbations. These orbits fill out what are called KAM tori.

An amazing extension of this result was started with the work of John Greene in 1968. He showed that if one continues to perturb a KAM torus, it reaches a stage where the nearby phase space [2.4] becomes self-similar (has fractal structure [3.2]). At this point the torus is "critical," and any increase in the perturbation destroys it. In a remarkable sequence of papers, Aubry and Mather showed that there are still quasiperiodic orbits that exist beyond this point, but instead of tori they cover cantor sets [3.5]. Percival actually discovered these for an example in 1979 and named them "cantori." Mathematicians tend to call them "Aubry-Mather" sets. These play an important role in limiting the rate of transport through chaotic regions.

Thus, the transition to chaos in Hamiltonian systems can be thought of as the destruction of invariant tori, and the creation of cantori. Chirikov was the first to realize that this transition to "global chaos" was an important physical phenomena. Local chaos also occurs in Hamiltonian systems (in the regions between the KAM tori), and is caused by the intersection of stable and unstable manifolds in what Poincaré called the "homoclinic trellis."

To learn more: See the introductory article by Berry, the text by Percival and Richards and the collection of articles on Hamiltonian systems by MacKay and Meiss [4.1]. There are a number of excellent advanced texts on Hamiltonian dynamics, some of which are listed in [4.1], but we also mention

Meyer, K. R. and G. R. Hall (1992), Introduction to Hamiltonian dynamical systems and the N-body problem (New York, Springer-Verlag).

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[4] To Learn More

[4.1] What should I read to learn more?

Popularizations

1 Gleick, J. (1987). Chaos, the Making of a New Science. London, Heinemann. http://www.around.com/chaos.html

2 Stewart, I. (1989). Does God Play Dice? Cambridge, Blackwell. http://www.amazon.com/exec/obidos/ASIN/1557861064

3 Devaney, R. L. (1990). Chaos, Fractals, and Dynamics: Computer Experiments in Mathematics. Menlo Park, Addison-Wesley http://www.amazon.com/exec/obidos/ASIN/1878310097

4 Lorenz, E., (1994) The Essence of Chaos, Univ. of Washington Press. http://www.amazon.com/exec/obidos/ASIN/0295975148

5 Schroeder, M. (1991) Fractals, Chaos, Power: Minutes from an infinite paradise W. H. Freeman New York:

Introductory Texts

1 Abraham, R. H. and C. D. Shaw (1992) Dynamics: The Geometry of Behavior, 2nd ed. Redwood City, Addison-Wesley.

2 Baker, G. L. and J. P. Gollub (1990). Chaotic Dynamics. Cambridge, Cambridge Univ. Press. http://www.cup.org/titles/catalogue.asp?isbn=0521471060

3 Devaney, R. L. (1986). An Introduction to Chaotic Dynamical Systems. Menlo Park, Benjamin/Cummings. http://math.bu.edu/people/bob/books.html

4 Kaplan, D. and L. Glass (1995). Understanding Nonlinear Dynamics, Springer-Verlag New York. http://www.cnd.mcgill.ca/books_understanding.html

5 Glendinning, P. (1994). Stability, Instability and Chaos. Cambridge, Cambridge Univ Press. http://www.cup.org/Titles/415/0521415535.html

6 Jurgens, H., H.-O. Peitgen, et al. (1993). Chaos and Fractals: New Frontiers of Science. New York, Springer Verlag. http://www.springer-ny.com/detail.tpl?isbn=0387979034

7 Moon, F. C. (1992). Chaotic and Fractal Dynamics. New York, John Wiley. http://www.wiley.com/WileyCDA/WileyTitle/productCd-0471545716.html

8 Percival, I. C. and D. Richard (1982). Introduction to Dynamics. Cambridge, Cambridge Univ. Press. http://www.cup.org/titles/catalogue.asp?isbn=0521281490

9 Scott, A. (1999). NONLINEAR SCIENCE: Emergence and Dynamics of Coherent Structures, Oxford http://www4.oup.co.uk/isbn/0-19-850107-2 http://www.imm.dtu.dk/documents/users/acs/BOOK1.html

10 Smith, P (1998) Explaining Chaos, Cambridge http://us.cambridge.org/titles/catalogue.asp?isbn=0521477476

11 Strogatz, S. (1994). Nonlinear Dynamics and Chaos. Reading, Addison-Wesley http://www.perseusbooksgroup.com/perseus-cgi-bin/display/0-7382-0453-6

12 Thompson, J. M. T. and H. B. Stewart (1986) Nonlinear Dynamics and Chaos. Chichester, John Wiley and Sons. http://www.wiley.com/WileyCDA/WileyTitle/productCd-0471876844.html

13 Tufillaro, N., T. Abbott, et al. (1992). An Experimental Approach to Nonlinear Dynamics and Chaos. Redwood City, Addison-Wesley. http://www.amazon.com/exec/obidos/ASIN/0201554410/

14 Turcotte, Donald L. (1992). Fractals and Chaos in Geology and Geophysics, Cambridge Univ. Press. http://www.cup.org/titles/catalogue.asp?isbn=0521567335

Introductory Articles

1 May, R. M. (1986). "When Two and Two Do Not Make Four." Proc. Royal Soc. B228: 241.

2 Berry, M. V. (1981). "Regularity and Chaos in Classical Mechanics, Illustrated by Three Deformations of a Circular Billiard." Eur. J. Phys. 2: 91-102.

3 Crawford, J. D. (1991). "Introduction to Bifurcation Theory." Reviews of Modern Physics 63(4): 991-1038.

3 Shinbrot, T., C. Grebogi, et al. (1992). "Chaos in a Double Pendulum." Am. J. Phys 60: 491-499.

5 David Ruelle. (1980). "Strange Attractors," The Mathematical Intelligencer 2: 126-37.

Advanced Texts

1 Arnold, V. I. (1978). Mathematical Methods of Classical Mechanics. New York, Springer. http://www.springer-ny.com/detail.tpl?isbn=038796890

2 Arrowsmith, D. K. and C. M. Place (1990), An Introduction to Dynamical Systems. Cambridge, Cambridge University Press. http://us.cambridge.org/titles/catalogue.asp?isbn=0521316502

3 Guckenheimer, J. and P. Holmes (1983), Nonlinear Oscillations, Dynamical Systems, and Bifurcation of Vector Fields, Springer-Verlag New York.

4 Kantz, H., and T. Schreiber (1997). Nonlinear time series analysis. Cambridge, Cambridge University Press http://www.mpipks-dresden.mpg.de/~schreibe/myrefs/book.html

5 Katok, A. and B. Hasselblatt (1995), Introduction to the Modern Theory of Dynamical Systems, Cambridge, Cambridge Univ. Press. http://titles.cambridge.org/catalogue.asp?isbn=0521575575

6 Hilborn, R. (1994), Chaos and Nonlinear Dyanamics: an Introduction for Scientists and Engineers, Oxford Univesity Press. http://www4.oup.co.uk/isbn/0-19-850723-2

7 Lichtenberg, A.J. and M. A. Lieberman (1983), Regular and Chaotic Motion, Springer-Verlag, New York .

8 Lind, D. and Marcus, B. (1995) An Introduction to Symbolic Dynamics and Coding, Cambridge University Press, Cambridge http://www.math.washington.edu/SymbolicDynamics/

9 MacKay, R.S and J.D. Meiss (eds) (1987), Hamiltonian Dynamical Systems A reprint selection, , Adam Hilger, Bristol

10 Nayfeh, A.H. and B. Balachandran (1995), Applied Nonlinear Dynamics: Analytical, Computational and Experimental Methods John Wiley& Sons Inc., New York http://www.wiley.com/WileyCDA/WileyTitle/productCd-0471593486.html

11 Ott, E. (1993). Chaos in Dynamical Systems. Cambridge University Press, Cambridge. http://us.cambridge.org/titles/catalogue.asp?isbn=0521010845

12 L.E. Reichl, (1992), The Transition to Chaos, in Conservative and Classical Systems: Quantum Manifestations Springer-Verlag, New York

13 Robinson, C. (1999), Dynamical Systems: Stability, Symbolic Dynamics, and Chaos, 2nd Edition, Boca Raton, CRC Press. http://www.crcpress.com/shopping_cart/products/product_detail.asp?sku=8495

14 Ruelle, D. (1989), Elements of Differentiable Dynamics and Bifurcation Theory, Academic Press Inc.

15 Tabor, M. (1989), Chaos and Integrability in Nonlinear Dynamics: an Introduction, Wiley, New York. http://www.wiley.com/WileyCDA/WileyTitle/productCd-0471827282.html
16 Wiggins, S. (1990), Introduction to Applied Nonlinear Dynamical Systems and Chaos, Springer-Verlag New York.
17 Wiggins, S. (1988), Global Bifurcations and Chaos, Springer-Verlag New York.

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[4.2] What technical journals have nonlinear science articles?

Physica D	The premier journal in Applied Nonlinear Dynamics
Nonlinearity	Good mix, with a mathematical bias
Chaos	AIP Journal, with a good physical bent
SIAM J. of Dynamical Systems	Online Journal with multimedia http://www.siam.org/journals/siads/siads.htm
Chaos Solitons and Fractals	Low quality, some good applications
Communications in Math Phys	an occasional paper on dynamics
Comm. in Nonlinear Sci. and Num. Sim.	New Elsevier journal http://www.elsevier.com/locate/cnsns
Ergodic Theory and Dynamical Systems	Rigorous mathematics, and careful work
International J of Bifurcation and Chaos	lots of color pictures, variable quality.
J Differential Equations	A premier journal, but very mathematical
J Dynamics and Diff. Eq.	Good, more focused version of the above
J Dynamics and Stability of Systems	Focused on Eng. applications. New editorial board--stay tuned.
J Fluid Mechanics	Some expt. papers, e.g. transition to turbulence
J Nonlinear Science	a newer journal--haven't read enough yet.
J Statistical Physics	Used to contain seminal dynamical systems papers
Nonlinear Dynamics	Haven't read enough to form an opinion
Nonlinear Science Today	Weekly News: http://www.springer-ny.com/nst/
Nonlinear Processes in Geophysics	New, variable quality...may be improving
Physics Letters A	Has a good nonlinear science section
Physical Review E	Lots of Physics articles with nonlinear emphasis
Regular and Chaotic Dynamics	Russian Journal http://web.uni.udm.ru/~rcd/

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[4.3] What are net sites for nonlinear science materials?

Bibliography

http://www.uni-mainz.de/FB/Physik/Chaos/chaosbib.html Mainz http site
ftp://ftp.uni-mainz.de/pub/chaos/chaosbib/ Mainz ftp site
http://www-chaos.umd.edu/publications/searchbib.html Seach the Mainz Site
http://www-chaos.umd.edu/publications/references.html Maryland
http://www.cpm.mmu.ac.uk/~bruce/combib/ Complexity Bibliography
http://www.mth.uea.ac.uk/~h720/research/ Ergodic Theory and Dynamical Systems
http://www.drchaos.net/drchaos/intro.html Nonlinear Dynamics Resources (pdf file)
http://www.nonlin.tu-muenchen.de/chaos/Projects/miguelbib Sanjuan's Bibliography

Preprint Archives

http://www.math.sunysb.edu/dynamics/preprints/ StonyBrook
http://cnls.lanl.gov/People/nbt/intro.html Los Alamos Preprint Server
http://xxx.lanl.gov/ Nonlinear Science Eprint Server
http://www.ma.utexas.edu/mp_arc/mp_arc-home.html Math-Physics Archive
http://www.ams.org/global-preprints/ AMS Preprint Servers List

Conference Announcements
http://at.yorku.ca/amca/conferen.htm Mathematics Conference List
http://www.math.sunysb.edu/dynamics/conferences/conferences.html StonyBrook List
http://www.nonlin.tu-muenchen.de/chaos/termine.html Munich List
http://xxx.lanl.gov/Announce/Conference/ Los Alamos List
http://www.tam.uiuc.edu/Events/conferences.html Theoretical & Applied Mechanics
http://www.siam.org/meetings/ds99/index.htm SIAM Dynamical Systems 1999

Newsletters
gopher://gopher.siam.org:70/11/siag/ds SIAM Dynamical Systems Group
http://www.amsta.leeds.ac.uk/Applied/news.dir/ UK Nonlinear News

Education Sites
http://math.bu.edu/DYSYS/ Devaney's Dynamical Systems Project

Electronic Journals
http://www.springer-ny.com/nst/ Nonlinear Science Today
http://www3.interscience.wiley.com/cgi-bin/jtoc?ID=38804 Complexity
http://journal-ci.csse.monash.edu.au/ Complexity International Journal

Electronic Texts
http://cnls.lanl.gov/People/nbt//Book/node1.html An experimental approach to nonlinear dynamics and chaos
http://www.nbi.dk/~predrag/QCcourse/ Lecture Notes on Periodic Orbits
http://hypertextbook.com/chaos/ The Chaos HyperTextBook

Institutes and Academic Programs
http://physicsweb.org/resources/dsearch.phtml Physics Institutes
http://ip-service.com/WiW/institutes.html Nonlinear Groups
http://www-chaos.engr.utk.edu/related.html Research Groups in Chaos

Java Applets Sites
http://physics.hallym.ac.kr/education/TIPTOP/VLAB/about.html Virtual Laboratory
http://monet.physik.unibas.ch/~elmer/pendulum/ Java Pendulum
http://kogs-www.informatik.uni-hamburg.de/~wiemker/applets/fastfrac/fastfrac.html Java Fractal Explorer
http://www.apmaths.uwo.ca/~bfraser/index.html B. Fraser's Nonlinear Lab
http://www.cmp.caltech.edu/~mcc/Chaos_Course/ Mike Cross' Demos

Who is Who in Nonlinear Dynamics
http://www.chaos-gruppe.de/wiw/wiw.html Munich List
http://www.math.sunysb.edu/dynamics/people/list.html Stonybrook List

Lists of Nonlinear sites
http://makeashorterlink.com/?C58C23C16 Netscape's List
http://cnls.lanl.gov/People/nbt/sites.html Tufillaro's List
http://cires.colorado.edu/people/peckham.scott/chaos.html Peckham's List
http://members.tripod.com/~IgorIvanov/physics/nonlinear.html Physics Encyclopedia

http://www.maths.ex.ac.uk/~hinke/dss/index.html Osinga's Software List

Dynamical Systems

http://www.math.sunysb.edu/dynamics/ Dynamical Systems Home Page
http://www.math.psu.edu/gunesch/entropy.html Entropy and Dynamics

Chaos sites

http://www.industrialstreet.net/chaosmetalink/ Chaos Metalink
http://bofh.priv.at/ifs/ Iterated Function Systems Playground
http://www.xahlee.org/PageTwo_dir/more.html Xah Lee's dynamics and Fractals pages
http://acl2.physics.gatech.edu/tutorial/outline.htm Tutorial on Control of Chaos
http://www.mathsoft.com/mathresources/constants/wellknown/article/0,,2090,00.html
All about Feigenbaum Constants
http://www.stud.ntnu.no/~berland/math/feigenbaum/ The Feigenbaum Fractal
http://members.aol.com/MTRw3/index.html Mike Rosenstein's Chaos Page.
http://www.vanderbilt.edu/AnS/psychology/cogsci/chaos/cspls.html Chaos in Psychology
http://www.eie.polyu.edu.hk/~cktse/NSR/ Movies and Demonstrations

Time Series

http://www.drchaos.net/drchaos/refs.html Dynamics and Time Series
http://astro.uni-tuebingen.de/groups/time/ Time series Analysis
http://www-personal.buseco.monash.edu.au/~hyndman/TSDL/index.htm
Time Series Data Library

Complex Systems Sites

http://www.math.upatras.gr/~mboudour/nonlin.html Complexity Home Page
http://www.calresco.org/ The Complexity & Artificial Life Web Site
http://www.physionet.org/ Complexity and Physiology Site

Fractals Sites

http://forum.swarthmore.edu/advanced/robertd/index.html#frac A Fractal Gallery
http://spanky.triumf.ca/www/welcome1.html The Spanky Fractal DataBase
http://sprott.physics.wisc.edu/fractals.htm Sprott's Fractal Gallery
http://fractales.inria.fr/ Projet Fractales
http://force.stwing.upenn.edu/~lau/fractal.html Lau's Fractal Stuff
http://skal.planet-d.net/quat/f_gal.html 3D Fractals
http://www.cnam.fr/fractals.html Fractal Gallery
http://www.fractaldomains.com/ Fractal Domains Gallery
http://home1.swipnet.se/~w-17723/fracpro.html Fractal Programs
http://xahlee.org/PageTwo_dir/MathPrograms_dir/mathPrograms.html#Fractals
Fractal Programs

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[5] Computational Resources

[5.1] What are general computational resources?

CAIN Europe Archives
http://www.can.nl/education/material/software.html Software Area
FAQ guide to packages from sci.math.num-analysis
ftp://rtfm.mit.edu/pub/usenet/news.answers/num-analysis/faq/part1
NIST Guide to Available Mathematical Software
http://gams.cam.nist.gov/
Mathematics Archives Software
http://archives.math.utk.edu/software.html
Matpack, C++ numerical methods and data analysis library
http://www.matpack.de/
Numerical Recipes Home Page
http://www.nr.com/

[5.2] Where can I find specialized programs for nonlinear science?

The Academic Software Library:

Chaos Simulations

Bessoir, T., and A. Wolf, 1990. Demonstrates logistic map, Lyapunov exponents, billiards in a stadi um, sensitive dependence, n-body gravitational motion.

Chaos Data Analyser

A PC program for analyzing time series. By Sprott, J.C. and G. Rowlands.
For more info:http://sprott.physics.wisc.edu/cda.htm

Chaos Demonstrations

A PC program for demonstrating chaos, fractals, cellular automata, and related nonlinear phenomen a. By J. C. Sprott and G. Rowlands.
System: IBM PC or compatible with at least 512K of memory.
Available: The Academic Software Library, (800) 955-TASL. $70.

Chaotic Dynamics Workbench

Performs interactive numerical experiments on systems modeled by ordinary differential equations, including: four versions of driven Duffing oscillators, pendulum, Lorenz, driven Van der Pol osc., d riven Brusselator, and the Henon-Heils system. By R. Rollins.
System: IBM PC or compatible, 512 KB memory.
Available: The Academic Software Library, (800) 955-TASL, $70

Applied Chaos Tools

Software package for time series analysis based on the UCSD group's, work. This package is a co mpanion for Abarbanel's book Analysis of Observed Chaotic Data, Springer-Verlag.
System: Unix-Motif, Windows 95/NT
For more info see: http://www.zweb.com/apnonlin/csp.html

AUTO

Bifurcation/Continuation Software (THE standard). The latest version is AUTO97. The GUI requir es X and Motif to be present. There is also a command line version AUTO86. The software is trans ported as a compressed file called auto.tar.Z.
System: versions to run under X windows--SUN or sgi or LINUX
Available: anonymous ftp from ftp://ftp.cs.concordia.ca/pub/doedel/auto

BZphase

Models Belousov- Zhabotinsky reaction based on the scheme of Ruoff and Noyes. The dynamics r

anges from simple quasisinusoidal oscillations to quasiperiodic, bursting, complex periodic and chaotic.
System: DOS 6 and higher + PMODE/W DOS Extender. Also openGL version
Available: http://members.tripod.com/~RedAndr/BZPhase.htm

Chaos
Visual simulation in two- and three-dimensional phase space; based on visual algorithms rather than canned numerical algorithms; well-suited for educational use; comes with tutorial exercises. By Bruce Stewart
System: Silicon Graphics workstations, IBM RISC workstations with GL
Available: http://msg.das.bnl.gov/~bstewart/software.html

Chaos
A Program Collection for the PC by Korsch, H.J. and H-J. Jodl, 1994, A book/disk combo that gives a hands-on, computer experiment approach to learning nonlinear dynamics. Some of the modules cover billiard systems, double pendulum, Duffing oscillator, 1D iterative maps, an "electronic chaos-generator", the Mandelbrot set, and ODEs.
System: IBM PC or compatible.
Available: $$http://www.springer-ny.com/catalog/np/updates/0-387-57457-3.html

CHAOS II
Chaos Programs to go with Baker, G. L. and J. P. Gollub (1990) Chaotic Dynamics. Cambridge, Cambridge Univ. http://www.cup.org/titles/catalogue.asp?isbn=0521471060
System: IBM, 512K memory, CGA or EGA graphics, True Basic
For more info: contact Gregory Baker, P.O. Box 278 ,Bryn Athyn, PA, 19009

Chaos Analyser
Programs to Time delay embedding, Attractor (3d) viewing and animation, Poincaré sections, Mutual information, Singular Value Decomposition embedding, Full Lyapunov spectra (with noise cancellation), Local SVD analysis (for determining the systems dimension). By Mike Banbrook.
System: Unix, X windows
For more info: http://www.ee.ed.ac.uk/~mb/analysis_progs.html

Chaos Cookbook
These programs go with J. Pritchard's book, The Chaos Cookbook System: Programs written in Visual Basic & Turbo Pascal
Available: $$http://www.amazon.com/exec/obidos/ASIN/0750617772

Chaos Plot
ChaosPlot is a simple program which plots the chaotic behavior of a damped, driven anharmonic oscillator.
System: Macintosh
For more info: http://archives.math.utk.edu/software/mac/diffEquations/.directory.html

Cubic Oscillator Explorer
The CUBIC OSCILLATOR EXPLORER is a Macintosh application which allows interactive exploration of the chaotic processes of the Cubic Oscillator, i.e..Duffing's equation.
System: Macintosh + Digidesign DSP Card, Digisystem init 2.6 and (optional) MIDI Manager
Available: (Missing??) Fractal Music

DataPlore
Signal and time series analysis package. Contains standard facilities for signal processing as well as advanced features like wavelet techniques and methods of nonlinear dynamics.

Systems: MS Windows, Linux, SUN Solaris 2.6
Available: $$http://www.datan.de/dataplore/

dstool

Free software from Guckenheimer's group at Cornell; DSTool has lots of examples of chaotic syste ms, Poincaré sections, bifurcation diagrams.
System: Unix, X windows.
Available: ftp://cam.cornell.edu/pub/dstool/

Dynamical Software Pro

Analyze non-linear dynamics and chaos. Includes ODEs, delay differential equations, discrete maps , numerical integration, time series embedding, etc.
System: DOS. Microsoft Fortran compiler for user defined equations.
Available: SciTech http://www.scitechint.com/

Dynamics: Numerical Explorations.

A book + disk by H. Nusse, and J.Yorke. A hands on approach to learning the concepts and the m any aspects in computing relevant quantities in chaos
System: PC-compatible computer or X-windows system on Unix computers
Available: $$ http://www.springer-ny.com/detail.tpl?isbn=0387982647

Dynamics Solver

Dynamics Solver solve numerically both initial-value problems and boundary-value problems for co ntinuous and discrete dynamical systems.
System: Windows 3.1 or Windows 95/98/NT
Available: http://tp.lc.ehu.es/jma/ds/ds.html

DynaSys

Phase plane portraits of 2D ODEs by Etienne Dupuis
System: Windows 95/98
Available: (Missing??)

FD3

A program to estimate fractal dimensions of a set. By DiFalco/Sarraille
System: C source code, suitable for compiling for use on a Unix or DOS platform.
Available: ftp://ftp.cs.csustan.edu/pub/fd3/

FracGen

FracGen is a freeware program to create fractal images using Iterated Function Systems. A tutorial is provided with the program. By Patrick Bangert
System: PC-compatible computer, Windows 3.1
Available: http://212.201.48.1/pbangert/site/fracgen.html

Fractal Domains

Generates of Mandelbrot and Julia sets. By Dennis C. De Mars
System: Power Macintosh
Available: http://www.fractaldomains.com/

Fractal Explorer

Generates Mandelbrot and Newton's method fractals. By Peter Stone
System: Power Macintosh
Available: http://usrwww.mpx.com.au/~peterstone/index.html

GNU Plotutils

The GNU plotutils package contains C/C++ function library for exporting 2-D vector graphics in many file formats, and for doing vector graphics animations. The package also contains several com mand-line programs for plotting scientific data, such as GNU graph, which is based on libplot, and ODE integration software.

System: GNU/Linux, FreeBSD, and Unix systems.

Available: http://www.gnu.org/software/plotutils/plotutils.html

Ilya

A program to visually study a reaction-diffusion model based on the Brusselator from Future Skills Software, Herber Sauro.

System: Requires Windows 95, at least 256 colours

Available : http://www.fssc.demon.co.uk/rdiffusion/ilya.htm

INSITE

(It's a Nonlinear Systems Investigative Toolkit for Everyone) is a collection for the simulation and c haracterization of dynamical systems, with an emphasis on chaotic systems. Companion software fo r T.S. Parker and L.O. Chua (1989) Practical Numerical Algorithms for Chaotic Systems Springer Verlag. See their paper "INSITE A Software Toolkit for the Analysis of Nonlinear Dynamical Syst ems," Proc. of the IEEE, **75**, 1081-1089 (1987).

System: C codes in Unix Tar or DOS format (later requires QuickWindowC
or MetaWINDOW/Plus 3.7C. and MS C compiler 5.1)

Available: INSITE SOFTWARE, p.o. Box 9662, Berkeley, CA , U.S.A.

Institut fur ComputerGraphik

A collection of programs for developing advanced visualization techniques in the field of three-dime nsional dynamical systems. By Löffelmann H., Gröller E.

System: various, requires AVS

Available: http://www.cg.tuwien.ac.at/research/vis/dynsys/

KAOS1D

A tool for studying one-dimensional (1D) discrete dynamical systems. Does bifurcation diagrams, etc. for a number of maps

System: PC compatible computer, DOS, VGA graphics

Available: http://www.if.ufrgs.br/~arenzon/jsoftw.html

LOCBIF

An interactive tool for bifurcation analysis of non-linear ordinary differential equations ODE's and maps**.** By Khibnik, Nikolaev, Kuznetsov and V. Levitin

System: Now part of XPP (See below)

Available: http://www.math.pitt.edu/~bard/classes/wppdoc/locbif.html

Lyapunov Exponents

Keith Briggs Fortran codes for Lyapunov exponents

System: any with a Fortran compiler

Available: http://more.btexact.com/people/briggsk2/

Lyapunov Exponents and Time Series

Based on Alan Wolf's algorithm, see [2.11], but a more efficient version.

System: Comes as C source, Fortran source, PC executable, etc

Available: http://www.cooper.edu/engineering/physics/wolf/ (Seems to be missing?)

Lyapunov Exponents and Time Series

Michael Banbrook's C codes for Lyapunov exponents & time series analysis

System: Sun with X windows.
Available: http://www.see.ed.ac.uk/~mb/analysis_progs.html

Lyapunov Exponents Toolbox (LET)

A user-contributed MATLAB toolbox that provides a graphical user interface for users to determin e the full sets of Lyapunov exponents and Lyapunov dimensions of discrete and continuous chaotic systems.
System: MATLAB 5
Available: ftp://ftp.mathworks.com/pub/contrib/v5/misc/let

Lyapunov.m

A Matlab program based on the QR Method , by von Bremen, Udwadia, and Proskurowski, Physica D, vol. **101**, 1-16, (1997)
System: Matlab
Available: http://www.usc.edu/dept/engineering/mecheng/DynCon/

Macintosh Dynamics Programs

Lists available at: http://hypertextbook.com/chaos/92.shtml
and http://www.xahlee.org/PageTwo_dir/MathPrograms_dir/mathPrograms.html

MacMath

Comes on a disk with the book MacMath, by Hubbard and West. A collection of programs for dyn amical systems (1 & 2 D maps, 1 to 3D flows). Version 9.2 is the current version, but West is work ing on a much improved update.
System: Macintosh
For more info: http://www.math.hmc.edu/codee/solvers/mac-math.html
Available: $$ Springer-Verlag http://www.springer-ny.com/detail.tpl?isbn=0387941355

Madonna

Solves Differential and Difference Equations. Runs STELLA. Has a parser with a control language. By Robert Macey and George Oster at Berkeley
System: Macintosh or Windows 95 or later
Available : $$ http://www.berkeleymadonna.com/

MatLab Chaos

A collection of routines for generate diagrams which illustrate chaotic behavior associated with the l ogistic equation.
System: Requires MatLab.
Available : ftp://ftp.mathworks.com/pub/contrib/misc/chaos/

MTRChaos

MTRCHAOS and MTRLYAP compute correlation dimension and largest Lyapunov exponents, del ay portraits. By Mike Rosenstein.
System: PC-compatible computer running DOS 3.1 or higher, 640K RAM, and EGA display. VGA & coprocessor recommended
Available: ftp://spanky.triumf.ca/pub/fractals/programs/ibmpc/

Nonlinear Dynamics Toolbox

Josh Reiss' NDT includes routines for the analysis of chaotic data, such as power spectral analyses, determination of the Lyapunov spectrum, mutual information function, prediction, noise reduction, and dimensional analysis.
System: Windows 95, 98, or NT

Available : Missing??

NLD Toolbox

This toolbox has many of the standard dynamical systems, By Jeff Brush
System: PC, MS-DOS.
Available: http://www.physik.tu-darmstadt.de/nlp/nldtools/nldtools.html

ODECalc

A program for integrating boundary value and initial value Problems for up to 9th order ODEs. By Optimal Designs.
System: PC 386+, DOS 3.3+, 16 bit arch.
Available : ftp://ftp.mecheng.asme.org/pub/EDU_TOOL/Ode200.exe

PHASER

Kocak, H., 1989. Differential and Difference Equations through Computer Experiments: with a sup plementary diskette containing PHASER: An Animator/Simulator for Dynamical Systems. Demons trates a large number of 1D-4D differential equations--many not chaotic--and 1D-3D difference eq uations.
System: PC-compatible
Available: Springer-Verlag http://www.springer-ny.com/detail.tpl?isbn=0387142029

PhysioToolkit

Software for physiologic signal processing and analysis, detection of physiologically significant events using both classical techniques and novel methods based on statistical physics and nonlinear dynamics
System: Unix
Available: http://www.physionet.org/physiotools/

Recurrence Quantification Analysis

Recurrence plots give a visual indication of deterministic behavior in complex time series. The progr am, by Webber and Zbilut creates the plots and quantifies the determinism with five measures.
System: DOS executable
Available:http://homepages.luc.edu/~cwebber/

SciLab

A simulation program similar in intent to MatLab. It's primarily designed for systems/signals work, and is large. From INRIA in France.
System: Unix, X Windows, 20 Meg Disk space.
Available : ftp://ftp.inria.fr/INRIA/Projects/Meta2/Scilab

StdMap

Iterates Area Preserving Maps, by J. D. Meiss. Iterates 8 different maps. It will find periodic orbits, cantori, stable and unstable manifolds, and allows you to iterate curves.
System: Macintosh
Available: http://amath.colorado.edu/faculty/jdm/stdmap.html

STELLA

Simulates dynamics for Biological and Social systems modelling. Uses a building block metaphor constructing models.
System: Macintosh and Windows PC
Available: $$ http://www.hps-inc.com/edu/stella/stella.htm

Time Series Tools
An extensive list of Unix tools for Time Series analysis
System: Unix
For more info: http://chuchi.df.uba.ar/guille/TS/tools/tools.html (Link down??)

Time Series Analysis from Darmstadt
Four prgrams Time Series analysis and Dimension calculation from the Institute of Applied Physic s at Darmstadt.
System: OS2 or Solaris/Linux/Win9X/NT + Fortran source
For more info: http://www.physik.tu-darmstadt.de/nlp/distribution.html

Time Series Analysis from Kennel
The program mkball finds the minimum embedding dimension using the false strands enhancemen t of the false neighbors algorithm of Kennel & Abarbanel.
System: any C compiler
Available: ftp://lyapunov.ucsd.edu/pub/nonlinear/mbkall.tar.gz

TISEAN Time Series Analysis
Agorithms for data representation, prediction, noise reduction, dimension and Lyapunov estimation, and nonlinearity testing. By Rainer Hegger, Holger Kantz and Thomas Schreiber
System: C, C++ and Fortran Codes for Unix,
Available: http://www.mpipks-dresden.mpg.de/~tisean/

Tufillaro's Programs
From the book Nonlinear Dynamics and Chaos by Tufillaro, Abbot and Reilly (1992) (for a sample section see http://www.drchaos.net/drchaos/Book/node1.html). A collection of programs for the M acintosh.
System: Macintosh
Available: http://www.drchaos.net/drchaos/bb.html

Unified Life Models (ULM)
ULM, by Stephane Legendre, is a program to study population dynamics and more generally, discr ete dynamical systems. It models any species life cycle graph (matrix models) inter- and intra-speci fic competition (non linear systems), environmental stochasticity, demographic stochasticity (branch ing processes), and metapopulations, migrations (coupled systems).
System: PC/Windows 3.X
Available: from http://www.snv.jussieu.fr

Virtual Laboratory
Simulations of 2D active media by the Complex Systems Group at the Max Planck Inst. in Berlin.
System: Requires PV-Wave by Visual Numerics $$http://www.vni.com/products/wave/
Available: $$ http://w3.rz-berlin.mpg.de/~mik/oertzen/vlm/m_contents.htm

VRA (Visual Recurrence Analysis)
VRA is a software to display and Study the recurrence plots, first described by Eckmann, Oliffson Kamphorst And Ruelle in 1987. With RP, one can graphically detect hidden patterns and structural changes in data or see similarities in patterns across the time series under study. By Eugene Konon ov
Stystem: Windows 95
Available: http://pweb.netcom.com/~eugenek/download.html

Xphased
Phase 3D plane program for X-windows systems (for systems like Lorenz, Rossler). Plot, rotate in 3-d, Poincaré sections, etc. By Thomas P. Witelski

System: X-windows, Unix, SunOS 4 binary
Available: http://www.alumni.caltech.edu/~witelski/xphased.html

XPP-Aut

Differential equations and maps for x-windows systems. Links to Auto for bifurcation analysis. By Bard Ermentrout
System: X-windows, Binaries for many unix systems
Available : ftp://ftp.math.pitt.edu/pub/bardware/tut/start.html

XSpiral

Simulate pattern formation in 2-D excitable media (in particular 2 models, one of them the FitzHugh-Nagumo). By Flavio Fenton.
System: X-windows
Available : (Missing??)

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