FIRST-ORDER SYSTEM $\mathcal{LL}^*$ (FOSLL*) FOR GENERAL SCALAR ELLIPTIC PROBLEMS IN THE PLANE[1]

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Abstract. This paper develops new first-order system $\mathcal{LL}^*$ (FOSLL*) formulations for scalar elliptic partial differential equations. It extends the work of [12], where the FOSLL* methodology was first introduced. One focus of that paper was to develop FOSLL* formulations that allow the use of H^1-conforming finite element spaces and optimal multigrid solution techniques to construct L^2 approximations of the dependent variables in the presence of discontinuous coefficients. The problems for which this goal was achieved were limited to those with no reaction term and with Dirichlet and Neumann boundaries that were individually connected, that is, each had at most one component. Here, new FOSLL* formulations are developed to achieve the same goals on a wider class of problems, including problems with reaction terms, Dirichlet and Neumann boundaries with multiple components, reentrant corners, and points at which Dirichlet and Neumann boundaries meet with an inner angle greater than $\pi/2$. The efficiency of the improved FOSLL* formulations is illustrated by a series of numerical examples.

Key words. first-order systems, least-squares methods, finite element methods, discontinuous coefficients, reduced regularity, non-smooth solutions

AMS subject classifications. 65N30

1. Introduction. First-order system $\mathcal{LL}^*$ (FOSLL*) was developed in an earlier paper [12] as a numerical method for solving partial differential equations (PDEs) that do not exhibit the regularity required by standard first-order system least squares (FOSLS [10, 11]). The purpose here is to extend the class of problems to which the FOSLL* approach can be efficiently applied. While we include a brief discussion of the context of this development below, the interested reader should consult [12] for more background and historical perspective.

Standard FOSLS recasts the original problem as an expanded first-order system, $Lu = f$, to which a least-squares principle is then applied. The usual goal is to reformulate the original problem as the minimization of a functional, $\|Lu - f\|^2$, whose bilinear part is equivalent to the product H^1 norm (i.e., the square root of the bilinear part is continuous and coercive in the norm formed by summing the H^1 norms applied to each dependent variable). This product H^1-equivalence means that the minimization process amounts to solving a weakly-coupled system of scalar elliptic equations, which, in turn, implies that H^1-conforming finite element spaces and multigrid solvers can be used to full efficiency. Unfortunately, standard FOSLS is product H^1-equivalent only under sufficient smoothness assumptions on the original problem (e.g., the domain, coefficients, and data). Inverse-norm versions of FOSLS could be used when the problem lacks sufficient smoothness, but these methods tend to lose efficiency, especially for problems with widely-varying coefficients.

Our purpose here is to continue the development of a potentially more efficient alternative, FOSLL* . As with FOSLS , the FOSLL* approach begins by recasting the original problem as an expanded first-order system, $Lu = f$. Now, however, instead of applying a least-squares principle to this *primal* problem, we introduce the *dual normal equations*, $LL^*w = f$, defined in terms of dual variable w and adjoint L^*. Note that $f = Lu$ so that $LL^*w = Lu$, which are the normal equations for the dual problem, $L^*w = u$. The original problem can now be recast as one of minimizing the functional, $\|L^*w - u\|^2$, which has the same minimizer as the functional $\|L^*w\|^2 - 2\langle w\,, f\rangle$.

If H^1-conforming finite element spaces are used in a standard FOSLS formulation, then it must fail when u is not in H^1. For this choice of finite element spaces, the discrete approximation produced

[1]This work was sponsored by the National Institute of Health under grant number 1–R01–EY12291–01, the National Science Foundation under grant number DMS–0084438, and the Department of Energy under grant numbers DE-FG03-94ER25217 and DE-FC02-01ER25479.

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by FOSLS cannot converge to the solution, u. It will, instead, converge optimally to the minimizer of $\|Lv - f\|^2$, that is, to $v \in H^1$ that minimizes $\|L(u - v)\|$. FOSLL* attempts to overcome this limitation by recasting the primal problem in terms of a dual variable, w, such that $L^*w = u$. The aim is to use L^* to lift the smoothness of u so that w is in H^1.

Consider the following scalar elliptic problem:

$$\begin{aligned} \nabla\cdot(A\nabla p) - \mathbf{b}\cdot\nabla p - cp &= f \text{ in } \Omega, \\ p &= 0 \text{ on } \Gamma_D, \\ \mathbf{n}\cdot A\nabla p &= 0 \text{ on } \Gamma_N, \end{aligned}$$

and define the flux variable $\mathbf{u} = \nabla p$ (for a complete list of assumptions, see (2.1) – (2.3)). One focus of the earlier paper [12] was to develop the FOSLL* methodology for problems of this type with the reduced regularity that arises by allowing discontinuous A. In that paper, the goal of using H^1-conforming finite element spaces to approximate the flux variable, $\mathbf{u}$, in the L^2 norm and the primal variable, p, in the H^1 norm was achieved through a two-stage procedure. The two-stage procedure described there is applicable only when $c = 0$ and when Γ_D and Γ_N each have at most one component.

The aim of this paper is to expand the class of problems for which H^1-conforming finite element spaces and optimal multigrid solvers can be efficiently used. In section 2, we show that when $c \neq 0$, the original FOSLL* formulation can be modified to achieve this goal, provided that the domain, Ω, is sufficiently smooth. By this we mean that the boundary of Ω contains no reentrant corners or corners at which $\overline{\Gamma}_D$ and $\overline{\Gamma}_N$ meet with an inner angle bigger than $\pi/2$. Such points are referred to as irregular boundary points. These are precisely the conditions under which $H(\nabla\cdot)\cap H(\nabla\times) \subset (H^1)^2$.

In section 3, we develop a new FOSLL* formulation that achieves the goal of allowing accurate approximation using H^1-conforming finite element spaces in the presence of irregular boundary points. The key idea behind this new approach is to first expand the domain of the primal problem in such a manner that the domain of the dual problem remains in a subspace of H^1. Generally, at this point the primal operator, L, is not bijective and the dual operator, L^*, is not surjective. The next step is to apply additional boundary conditions to the slack variables in the primal equations so that fewer boundary conditions are needed for the dual problem. The aim is that the primal operator, L, becomes bijective and the dual operator becomes surjective. This process generally means that the dual equations are not uniquely solvable. However, this is not an issue for the FOSLL* formulation, since any one solution of the dual problem, say, w, yields the primal solution, $L^*w = u$. This approach is limited to problems for which $\Gamma_D \neq \emptyset$. The pure Neumann case remains an open question. In section 3.2, we show that, in the case $\mathbf{b} = \mathbf{0}$ and either $c > 0$ or $c = 0$, the FOSLL* approximation is equivalent to a Galerkin formulation of the original boundary value problem, (2.1)-(2.3).

The numerical results presented in Section 4 confirm the optimality of H^1-conforming finite element spaces and multigrid solvers for the new FOSLL* formulation. The loss of unique solution for the dual problem is not an issue for the FOSLL* approximation in that we seek any dual solution for which $L^*w = u$. However, the loss of uniqueness does affect the multigrid solution algorithm. In section 4.2, we develop an additional modification that mitigates this effect. Section 5 contains conclusions.

Alternatives to the approach we develop here are described in detail in [1] and include adding H^1 singular basis functions in standard Galerkin methods to enhance the rate of convergence (c.f. [23],[14], [8] and [9]); the use of $H(\text{div})$-conforming finite element spaces with mixed formulations (see [7]) or with FOSLS functionals that are based on $H(\text{div})$ (see [10], [20], and [21]); and including $H(\nabla\cdot)\cap H(\nabla\times)$ singular functions in a FOSLS formulation ([1],[2]). Standard finite element spaces can be used with FOSLS functionals that are weighted to eliminate the overall impact on accuracy of the singular behavior of the flux ([14], [19], [24]). Alternatives similar to the FOSLL* formulation use FOSLS based on inverse norms ([3],[5], [6], [13]).

We begin in the next section with a brief overview of the current theory underlying FOSLS and FOSLL* as a way of exposing the need for modifications of the original FOSLL* approach.

2. General FOSLS and FOSLL* theory. This section summarizes the principles and theory underlying the FOSLS and FOSLL* methods. For more detail and historical perspective, see [10, 11, 12, 4]. The main goal of this section is to clarify the need for modifying the FOSLL* method introduced in [12].

2.1. Model problem. Let Ω be a bounded, open, simply connected domain in $\mathbb{R}^2$ with Lipschitz boundary, $\partial\Omega$. Let $\bigcup_{i=1}^{M}(\overline{\Gamma}_{D,i} \cup \overline{\Gamma}_{N,i}) = \partial\Omega$ be a partition of the boundary, interlaced so that every pair $(\Gamma_{D,i}, \Gamma_{D,i+1})$ is separated by a Neumann boundary segment $\Gamma_{N,i}$, and every Neumann pair is similarly separated by a Dirichlet segment. The Neumann and Dirichlet boundaries of the problem are defined by $\Gamma_N := \bigcup_{i=1}^{M} \Gamma_{N,i}$ and $\Gamma_D := \bigcup_{i=1}^{M} \Gamma_{D,i}$, respectively. Let $\mathbf{n}$ be the outward unit normal vector and $\mathbf{t}$ the counter-clockwise-oriented tangent vector on $\partial\Omega$. We do not consider the pure Neumann case here, so Γ_D is assumed to have positive measure.

The FOSLL* methodology has application in many contexts, including elliptic systems of partial differential equations. However, in this paper, we restrict our considerations to the following reaction-convection-diffusion boundary value problem (BVP):

$$\nabla\cdot(A\nabla p) - \mathbf{b}\cdot\nabla p - cp = f \text{ in } \Omega, \tag{2.1}$$
$$p = 0 \text{ on } \Gamma_D, \tag{2.2}$$
$$\mathbf{n}\cdot A\nabla p = 0 \text{ on } \Gamma_N, \tag{2.3}$$

where $f \in L^2(\Omega)$, $0 \le c \in L^\infty(\Omega)$, $\mathbf{b} \in L^\infty(\Omega) \cap H(\nabla\cdot)$, and $A(\mathbf{x})$ is a 2x2 matrix of $L^\infty(\Omega)$-functions that is uniformly symmetric positive definite, i.e., there exists $\lambda_1 \ge \lambda_0 > 0$ such that

$$\lambda_0 \boldsymbol{\xi}\cdot\boldsymbol{\xi} \le \boldsymbol{\xi}\cdot A(\mathbf{x})\boldsymbol{\xi} \le \lambda_1 \boldsymbol{\xi}\cdot\boldsymbol{\xi}$$

for all $\boldsymbol{\xi} \in \mathbb{R}^2$ and $\mathbf{x} \in \Omega$. We also assume that both (2.1)-(2.3) and the adjoint problem,

$$\nabla\cdot(A\nabla p) + \nabla\cdot(\mathbf{b}p) - cp = f \text{ in } \Omega, \tag{2.4}$$
$$p = 0 \text{ on } \Gamma_D, \tag{2.5}$$
$$\mathbf{n}\cdot(A\nabla p + \mathbf{b}p) = 0 \text{ on } \Gamma_N, \tag{2.6}$$

have unique solutions in $H^1(\Omega)$.

We make use of to the following standard differential operators:

$$\nabla s = \operatorname{grad} s = \begin{pmatrix} \partial_x s \\ \partial_y s \end{pmatrix}, \qquad \nabla\cdot\mathbf{v} = \operatorname{div}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \partial_x v_1 + \partial_y v_2,$$
$$\nabla^\perp s = \operatorname{rot} s = \begin{pmatrix} \partial_y s \\ -\partial_x s \end{pmatrix}, \qquad \nabla\times\mathbf{v} = \operatorname{curl}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = -\partial_y v_1 + \partial_x v_2.$$

We use $\langle\cdot,\cdot\rangle$ and $||\cdot||$ to denote the respective L^2 inner product and norm and $\mathcal{D}$, $\mathcal{R}$, and $\mathcal{N}$ for the respective domain, range, and null space of an operator. We also use $||\cdot||_1$ to denote the $H^1(\Omega)$ norm: $||s||_1^2 = ||s||^2 + ||\nabla s||^2$. As usual, norms of vectors are meant to be taken componentwise, so that $||\nabla s|| = (||\frac{\partial s}{\partial x}||^2 + ||\frac{\partial s}{\partial y}||^2)^{1/2}$, for example.

2.2. FOSLS . We begin with a brief introduction to the main ideas of FOSLS as a way of providing a foundation for the FOSLL* methodology. We describe how it works for domains without irregular boundary points and show why it may fail for domains with irregular boundary points.

Standard FOSLS transforms BVP (2.1)-(2.3) into a first-order system to which an L^2 norm minimization principle is applied. This transformation can be done by introducing the gradient,

$\mathbf{u} = \nabla p$, as an dependent variable and adding the curl constraint, $\nabla\times\mathbf{u} = 0$ on Ω, and tangential boundary condition, $\mathbf{t}\cdot\mathbf{u} = 0$ on Γ_D. The resulting first-order system then has the form

$$L_0(\mathbf{u},p) = (\mathbf{0},f,0)^t \text{ in } \Omega, \tag{2.7}$$
$$\mathbf{t}\cdot\mathbf{u} = 0 \text{ on } \Gamma_D, \tag{2.8}$$
$$\mathbf{n}\cdot A\mathbf{u} = 0 \text{ on } \Gamma_N, \tag{2.9}$$
$$p = 0 \text{ on } \Gamma_D, \tag{2.10}$$

where

$$L_0(\mathbf{u},p) := \begin{bmatrix} I & -\nabla \\ \nabla\cdot A - \mathbf{b}\cdot & -c \\ -\nabla\times & 0 \end{bmatrix} \begin{pmatrix} \mathbf{u} \\ p \end{pmatrix} = \begin{pmatrix} \mathbf{u} - \nabla p \\ \nabla\cdot A\mathbf{u} - \mathbf{b}\cdot\mathbf{u} - cp \\ -\nabla\times\mathbf{u} \end{pmatrix}. \tag{2.11}$$

The least-squares functional to be minimized is

$$\mathcal{F}_0(\mathbf{v},t) = \left\| L_0(\mathbf{v},t) - (\mathbf{0},f,0)^t \right\|^2.$$

Since we want this functional to exist for all $(\mathbf{v},t) \in \mathcal{D}(L_0)$, we need

$$\mathcal{R}(L_0) \subseteq \left(L^2(\Omega)\right)^4. \tag{2.12}$$

We are, thus, lead to choose

$$\mathcal{D}(L_0) = (H_N(\nabla\cdot A;\Omega) \cap H_D(\nabla\times;\Omega)) \times H^1_D(\Omega),$$

where, for a general 2x2 matrix B, we define

$$H^1_J(\Omega) := \left\{ s \in H^1(\Omega) : s = 0 \text{ on } \Gamma_J \right\},$$
$$H_J(\nabla\cdot B;\Omega) := \left\{ \mathbf{w} \in (L^2(\Omega))^2 : \nabla\cdot(B\mathbf{w}) \in L^2(\Omega), \mathbf{n}\cdot B\mathbf{w} = 0 \text{ on } \Gamma_J \right\},$$
$$H_J(\nabla\times B;\Omega) := \left\{ \mathbf{w} \in (L^2(\Omega))^2 : \nabla\times(B\mathbf{w}) \in L^2(\Omega), \mathbf{t}\cdot B\mathbf{n} = 0 \text{ on } \Gamma_J \right\},$$

for $J \in \{N,D\}$. Moreover, $H_J(\nabla\cdot;\Omega) := H_J(\nabla\cdot I;\Omega)$ and $H_J(\nabla\times;\Omega) := H_J(\nabla\times I;\Omega)$, where I is the 2×2 identity matrix. Clearly, $\mathcal{D}(L_0)$ is a Hilbert space under the norm

$$\|(\mathbf{v},t)\|^2_{L_0} := \|\mathbf{v}\|^2 + \|\nabla\cdot A\mathbf{v}\|^2 + \|\nabla\times\mathbf{v}\|^2 + \|t\|^2_1.$$

Since BVP (2.1)-(2.3) is well posed by assumption, we know that (2.7)-(2.10) has a unique solution in $\mathcal{D}_0$. Thus, $\mathcal{F}_0$ has a unique minimizer in $\mathcal{D}_0$ with minimum value 0. We minimize functional $\mathcal{F}_0$ in the weak sense, i.e., we look for solutions of the corresponding variational problem:
Find $(\mathbf{u},p) \in \mathcal{D}_0$ *such that*

$$\left\langle L_0(\mathbf{u},p) - (\mathbf{0},f,0)^t, L_0(\mathbf{v},t) \right\rangle = 0 \tag{2.13}$$

for all $(\mathbf{v},t) \in \mathcal{D}_0$.

A convenient choice for this FOSLS formulation is to discretize variational problem (2.13) using H^1-conforming finite element spaces, such as bilinears on quadrilaterals or linears on triangles. As the mesh size of the discretization tends to zero, the use of H^1-conforming finite element spaces yields converging approximations of the solution provided that solution is in H^1. This approach requires

$$(\mathbf{u},p) \in \left(H^1(\Omega)\right)^3 \tag{2.14}$$

for FOSLL* approximations using H^1-conforming finite elements to converge to the solution of primal problem (2.7)-(2.10). The requirement (2.14) is more restrictive than (2.12) and is, in general, not fulfilled for problems with irregular boundary points or discontinuous coefficient matrix A. In such cases, the FOSLS approximations, $(\mathbf{u}^h, p^h)$, do not converge to the solution of (2.7)-(2.10), as the following example illustrates.

EXAMPLE 2.1. *Define the following L-shaped domain:*

$$\Omega = \left\{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\|_\infty < 1 \text{ and } \theta(\mathbf{x}) \in (0, 3\pi/2)\right\}, \tag{2.15}$$

where $\theta = \arcsin(y/x)$. *Let* $A = I$, $c = 1$, *and* $\mathbf{b} = (-y/10, 10x)^t$. *The Neumann boundary consists of three parts,*

$$\Gamma_{N,1} = \{(x,y) \in \partial\Omega : x \in (0,1), y = 1\}, \quad \Gamma_{N,2} = \{(x,y) \in \partial\Omega : y = -1\}$$
$$\Gamma_{N,3} = \{(x,y) \in \partial\Omega : x = -1, y \in (0,1)\}.$$

The three remaining parts of $\partial\Omega$ *form* Γ_D. *This domain contains irregular boundary points at* $(0,0)$, $(-1,0)$, *and* $(0,1)$.

Let (r,θ) *denote standard polar coordinates on* $\mathbb{R}^2$ *and let*

$$p = \delta(r) r^{2/3} \sin(2\theta/3). \tag{2.16}$$

Then, p *is a solution of BVP (2.1)-(2.3) when* $f = \sin(2\theta/3) r^{2/3}(\delta''(r) + \frac{7}{3r}\delta'(r)) - \mathbf{b}\cdot\nabla p - cp$. *Here,* $\delta(r) \in C^2(\Omega)$ *is a cut-off function that satisfies* $\delta(r) = 1$ *for* $r < 1/4$ *and* $\delta(r) = 0$ *for* $r > 3/4$. *Clearly,* $f \in L^2(\Omega)$, *but* $\mathbf{u} = \nabla p$ *is not in* $(H^\alpha(\Omega))^2$ *for any* $\alpha \geq 2/3$. *Table 2.1 shows the results of numerical experiments for this problem with error norms* $||p - p^h||_0$ *and* $||\nabla p - \mathbf{u}^h||_0$ *for a sequence of uniform meshes with decreasing meshsizes* h *and standard bilinear* H^1*-conforming finite element spaces. Standard FOSLS clearly fails for this example. A closer look at* $\mathbf{u}^h$ *shows that the FOSLS approximation is completely unaware of the singularities in the gradient at the reentrant corner of* Ω.

h	1/4	1/8	1/16	1/32	1/64	1/128	1/256
$\|\|p - p^h\|\|_0$	.1461	.1521	.1543	.1550	.1551	.1550	.1549
$\|\|\nabla p - \mathbf{u}^h\|\|_0$	.8433	.8267	.8199	.8173	.8152	.8137	.8127

TABLE 2.1
Error norms for Example 2.1 on a sequence of uniform meshes with meshsizes h.

2.3. FOSLL* . The FOSLL* method was developed to overcome this difficulty with standard FOSLS , while continuing to use standard H^1-conforming finite element spaces in the discretization process. Clearly, H^1-conforming spaces cannot be used to approximate the nonsmooth primal solution, $(\mathbf{u}, p)$, so FOSLL* instead attempts to introduce a dual first-order system whose solution is in H^1.

The main idea can be motivated by looking at the simplest discrete analog, that is, a linear system of equations, $Ax = b$. Solving the corresponding least-squares problem of minimizing $\|Ax - b\|^2_{\ell^2}$ leads to the normal equations, $A^tAx = A^tb$, and the weak form, $\langle Ax, Az\rangle = \langle b, Az\rangle$ *for all* z. This is analogous to what FOSLS does at the PDE level. But another way to recast $Ax = b$ as a minimization problem is to recognize that if $Ax = b$ has a solution, then so does $AA^ty = b$. Note that this system for dual variable y is the normal equations for dual problem $A^ty = x$, and that it can be recast as the minimization of $\|A^ty - x\|^2_{\ell^2}$, which has the same minimizer as the functional $\langle A^ty, A^ty\rangle - 2\langle y, b\rangle$. This leads to the weak form, $\langle A^ty, A^tz\rangle = \langle b, z\rangle$ *for all* z. Note that $x = A^ty$ yields the minimal norm solution of the original problem, $Ax = b$. This idea is formally applicable at the PDE level since our primal problem surely has a solution.

While simpler approaches are possible in some cases, a fairly general methodology for applying FOSLL* is to attempt to reformulate the original BVP as a first-order primal problem whose associated operator, L_1, and adjoint, L_1^*, are bijective. This guarantees the existence of a unique solution for the dual normal equations, $L_1 L_1^* w = f$.

This bijectivity is achieved for BVP (2.1)-(2.3) by incorporating a scalar slack variable, q, into the system and using the scaled gradient, $\tilde{\mathbf{u}} = A^{1/2}\nabla p$. (Here, we incorporate a slightly different scaling than in (2.11) because it has computational advantages.) This is done in such a way that $(\tilde{\mathbf{u}}, p, 0)$ solves the primal problem, which for BVP (2.1)-(2.3) is given by

$$L_1(\tilde{\mathbf{u}}, p, q) = (\mathbf{0}, f, 0)^t \text{ in } \Omega, \tag{2.17}$$
$$\mathbf{t} \cdot A^{-1/2}\tilde{\mathbf{u}} = 0 \text{ on } \Gamma_D, \tag{2.18}$$
$$\mathbf{n} \cdot A^{1/2}\tilde{\mathbf{u}} = 0 \text{ on } \Gamma_N, \tag{2.19}$$
$$p = 0 \text{ on } \Gamma_D, \tag{2.20}$$
$$q = 0 \text{ on } \Gamma_N, \tag{2.21}$$

where

$$\begin{aligned} L_1(\tilde{\mathbf{u}}, p, q) &:= \begin{bmatrix} A^{-1/2} & -\nabla & -\nabla^\perp \\ \nabla\cdot A^{1/2} - \mathbf{b}\cdot A^{-1/2} & -c & 0 \\ -\nabla\times A^{-1/2} & 0 & 0 \end{bmatrix} \begin{pmatrix} \tilde{\mathbf{u}} \\ p \\ q \end{pmatrix} \\ &= \begin{pmatrix} A^{-1/2}\tilde{\mathbf{u}} - \nabla p - \nabla^\perp q \\ \nabla\cdot(A^{1/2}\tilde{\mathbf{u}}) - \mathbf{b}\cdot A^{-1/2}\tilde{\mathbf{u}} - cp \\ -\nabla\times(A^{-1/2}\tilde{\mathbf{u}}) \end{pmatrix}. \end{aligned} \tag{2.22}$$

The domain of L_1 is

$$\mathcal{D}(L_1) = \Big(H_N(\nabla\cdot A^{1/2}; \Omega) \cap H_D(\nabla\times A^{-1/2}; \Omega)\Big) \times H_D^1(\Omega) \times H_N^1(\Omega),$$

which is a Hilbert space under the norm

$$\|(\mathbf{v}, t, z)\|_{L_1}^2 := \|\mathbf{v}\|^2 + \left\|\nabla\cdot(A^{1/2}\mathbf{v})\right\|^2 + \left\|\nabla\times(A^{-1/2}\mathbf{v})\right\|^2 + \|t\|_1^2 + \|z\|_1^2 . \tag{2.23}$$

The FOSLL* approach is to approximate the solution, $(\mathbf{w}, r, s)$, of the corresponding dual problem,

$$L_1^*(\mathbf{w}, r, s) = (\tilde{\mathbf{u}}, p, q)^t = (A^{1/2}\nabla p, p, 0)^t \text{ in } \Omega, \tag{2.24}$$
$$\mathbf{t} \cdot A^{-1/2}\mathbf{w} = 0 \text{ on } \Gamma_D, \tag{2.25}$$
$$\mathbf{n} \cdot A^{1/2}\mathbf{w} = 0 \text{ on } \Gamma_N, \tag{2.26}$$
$$r = 0 \text{ on } \Gamma_D, \tag{2.27}$$
$$s = 0 \text{ on } \Gamma_N, \tag{2.28}$$

where the adjoint operator is defined by

$$L_1^*(\mathbf{w}, r, s) = \begin{bmatrix} A^{-1/2} & -A^{1/2}\nabla - A^{-1/2}\mathbf{b} & -A^{-1/2}\nabla^\perp \\ \nabla\cdot & -c & 0 \\ -\nabla\times & 0 & 0 \end{bmatrix} \begin{pmatrix} \mathbf{w} \\ r \\ s \end{pmatrix}. \tag{2.29}$$

The domain of L_1^* is

$$\mathcal{D}(L_1^*) = (H_N(\nabla\cdot; \Omega) \cap H_D(\nabla\times; \Omega)) \times H_D^1(\Omega) \times H_N^1(\Omega).$$

which is a Hilbert space under the norm

$$\|(\mathbf{v}, t, z)\|_{L_1^*}^2 := \|\mathbf{v}\|^2 + \|\nabla\cdot(\mathbf{v})\|^2 + \|\nabla\times(\mathbf{v})\|^2 + \|t\|_1^2 + \|z\|_1^2 . \tag{2.30}$$

This formulation is similar to the FOSLL$^*_e$ formulation described in [12]. The difference is that, in (2.29), the coefficient matrix, A, only appears outside of the differential operators. Note, also, that this scaling yields $A^{1/2}\nabla r$ orthogonal to $A^{-1/2}\nabla^\perp s$, which produces better performance for the multigrid solvers. A minor modification of the proof of Theorem 4.1 in [12] yields the following result.

THEOREM 2.2. *Operators L_1 and L_1^* are bijective from $\mathcal{D}(L_1)$ and $\mathcal{D}(L_1^*)$, respectively, onto $(L^2(\Omega))^4$. Further, L_1 and L_1^* are coercive and continuous in the norms defined in (2.23) and (2.30), respectively.*

Proof: The proof requires the assumption that the adjoint problem (2.4)-(2.6) is well posed and follows with minor modifications from the proof of Theorem 4.1 in [12], together with an application of Lemma 2.1 in [12].

□

Solving the dual problem is equivalent to minimizing the dual functional,

$$\mathcal{F}_1^*(\mathbf{v},t,z) = \left\| L_1^*(\mathbf{v},t,z) - (\tilde{\mathbf{u}},p,q)^t \right\|^2, \tag{2.31}$$

on $\mathcal{D}(L_1^*)$. The associated weak form is as follows:

Find $(\mathbf{w},r,s) \in \mathcal{D}(L_1^*)$ such that

$$\langle L_1^*(\mathbf{w},r,s), L_1^*(\mathbf{v},t,z)\rangle = \left\langle (\tilde{\mathbf{u}},p,q)^t, L_1^*(\mathbf{v},t,z)\right\rangle, \tag{2.32}$$

for all $(\mathbf{v},t,z) \in \mathcal{D}(L_1^*)$.

The unknown solution, $(\tilde{\mathbf{u}},p,q)$, is eliminated from the right side of (2.32) by rewriting the right side as follows:

$$\left\langle (\tilde{\mathbf{u}},p,q)^t, L_1^*(\mathbf{v},t,z)\right\rangle = \left\langle L_1(\tilde{\mathbf{u}},p,q), (\mathbf{v},t,z)^t\right\rangle = \left\langle (\mathbf{0},f,0)^t, (\mathbf{v},t,z)^t\right\rangle.$$

After discretizing this variational form and computing an approximation, $(\mathbf{w}^h,r^h,s^h)$, for the dual unknowns, an L^2 approximation, $(\tilde{\mathbf{u}}^h,p^h,q^h)$, for the primal unknowns is computed easily by applying the adjoint: $(\tilde{\mathbf{u}}^h,p^h,q^h)^t = L_1^*(\mathbf{w}^h,r^h,s^h)$.

This formulation of FOSLL* works well with H^1-conforming finite element spaces if the violation of the crucial regularity condition, (2.14), is due only to the discontinuities in A. This can be most easily seen by noting that in (2.29) the coefficients are never differentiated. However, in the presence of irregular boundary points, we may be left with the difficulty that

$$H_N(\nabla\cdot;\Omega) \cap H_D(\nabla\times;\Omega) \not\subset \left(H^1(\Omega)\right)^2. \tag{2.33}$$

For example, (2.33) holds if the boundary of Ω contains reentrant corners or points in $\overline{\Gamma_D} \cap \overline{\Gamma_N}$ with an inner angle bigger than $\pi/2$ (c.f. [17]). If H^1-conforming finite element spaces are used to approximate the solution to (2.32), then the approximation will not, in general, converge to the solution, but rather to the closest element in $(H^1)^4$ to the solution. In general, this error will not have local support. In the next section, we introduce a modification to FOSLL* that overcomes this difficulty.

We close this section by demonstrating numerically how the FOSLL* formulation described above fails in the presence of irregular boundary points.

EXAMPLE 2.3. *We apply the FOSLL* method (2.32), using H^1-conforming finite elements, to the BVP from Example 2.1. Table 2.2 shows that the L^2 norm of the errors of the approximations for p and $\tilde{\mathbf{u}}$ stagnate as h decreases.*

3. Improved FOSLL* . We begin here by introducing modifications to the standard FOSLL* formulation that overcome the shortcomings for problems with irregular boundary points. We then describe how the method can be made more efficient for the special cases $\mathbf{b} = \mathbf{0}$ and $c = 0$.

h	1/4	1/8	1/16	1/32	1/64	1/128	1/256
$\|\|p-p^h\|\|_0$	.0482	.0259	.0231	.0230	.0229	.0228	.0228
$\|\|\tilde{\mathbf{u}}-\tilde{\mathbf{u}}^h\|\|_0$	.6767	.3202	.1825	.1220	.1001	.0933	.0914

TABLE 2.2

Error norms for the standard FOSLL approximations for Example 2.3 on a sequence of uniform meshes with meshsizes h.*

As a starting point of our improvements, we revert to the scaling used in (2.11). While the scaling in (2.22) is preferable in practice, we use this simpler scaling for ease of exposition. All results in this section can be easily generalized to the scaling in (2.22).

Thus, we define the unscaled gradient, $\mathbf{u} = \nabla p$, as a dependent variable. The primal problem has the form $\mathcal{L}_0(\mathbf{u}, p, q) = (\mathbf{0}, f, 0)^t$, where q is a slack variable as introduced in the previous subsection,

$$\mathcal{L}_0 = \begin{bmatrix} I & -\nabla & -\nabla^\perp \\ (\nabla\cdot A - \mathbf{b}\cdot) & -c & 0 \\ -\nabla\times & 0 & -d \end{bmatrix}, \tag{3.1}$$

and d is a nonnegative analytic function on Ω.

Following the development for standard FOSLL* , the domain of $\mathcal{L}_0$ is given by

$$\mathcal{D}(\mathcal{L}_0) = (H_N(\nabla\cdot A;\Omega) \cap H_D(\nabla\times;\Omega)) \times H^1_D(\Omega) \times H^1_N(\Omega).$$

Clearly, $(\nabla p, p, 0)$ solves this problem when p is the solution of the BVP (2.1)-(2.3). The corresponding dual problem is

$$\mathcal{L}_0^*(\mathbf{w}, r, s) := \begin{bmatrix} I & -(A\nabla + \mathbf{b}) & -\nabla^\perp \\ \nabla\cdot & -c & 0 \\ -\nabla\times & 0 & -d \end{bmatrix} \begin{pmatrix} \mathbf{w} \\ r \\ s \end{pmatrix} = \begin{pmatrix} \mathbf{u} \\ p \\ q \end{pmatrix}, \tag{3.2}$$

on the adjoint domain

$$\mathcal{D}(\mathcal{L}_0^*) = (H_N(\nabla\cdot;\Omega) \cap H_D(\nabla\times;\Omega)) \times H^1_D(\Omega) \times H^1_N(\Omega).$$

Formulating the FOSLL* method using $\mathcal{L}_0^*$ reveals exactly the same difficulty as the formulation using L_1^* in the last subsection. While discontinuous coefficients do not cause difficulties, irregular boundary points do, because they imply $H_N(\nabla\cdot;\Omega) \cap H_D(\nabla\times;\Omega) \not\subset \left(H^1(\Omega)\right)^2$, which in turn implies $\mathcal{D}(\mathcal{L}_1^*) \not\subset \left(H^1(\Omega)\right)^4$.

The next step is to introduce a modified operator, $\mathcal{L}_1$, that is formally identical to $\mathcal{L}_0$ in (3.1), but has a different domain. The aim is to expand the domain of $\mathcal{L}_1$ so that the domain of its adjoint shrinks to a subspace of $\left(H^1(\Omega)\right)^4$. To see how this is done, note that the second and third entries in the first row of $\mathcal{L}_0$ in (3.1) can be rewritten as follows:

$$\begin{bmatrix} -\nabla & -\nabla^\perp \end{bmatrix} = \begin{bmatrix} -\partial_x & -\partial_y \\ -\partial_y & \partial_x \end{bmatrix} = \begin{bmatrix} -\nabla\cdot \\ \nabla\times \end{bmatrix}.$$

Thus, instead of asking the gradients of p and q to be in $L^2(\Omega)$ individually, we may impose the more general condition that the div and curl of the pair (p, q) be in $L^2(\Omega)$. We are thus lead to rewrite $\mathcal{L}_0$ as

$$\mathcal{L}_1 = \begin{bmatrix} I & -\nabla\cdot \\ & \nabla\times \\ (\nabla\cdot A - \mathbf{b}\cdot) & \\ -\nabla\times & -B \end{bmatrix}, \text{ where } B = \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix},$$

so that

$$\mathcal{D}(\mathcal{L}_1) = (H_N(\nabla\cdot A;\Omega) \cap H_D(\nabla\times;\Omega)) \times \mathcal{H}_{DN}(\Omega),$$

where

$$\mathcal{H}_{DN}(\Omega) := \{(v_1, v_2) \in H(\nabla\cdot;\Omega) \cap H(\nabla\times;\Omega) : v_1 = 0 \text{ on } \Gamma_D, v_2 = 0 \text{ on } \Gamma_N\}.$$

Integration by parts then shows that the domain of $\mathcal{L}_1^*$ is in $\left(H^1(\Omega)\right)^4$:

$$\mathcal{D}(\mathcal{L}_1^*) = \left(H_N(\nabla\cdot;\Omega) \cap H_D(\nabla\times;\Omega) \cap \left(H^1(\Omega)\right)^2\right) \times H_D^1(\Omega) \times H_N^1(\Omega).$$

Unfortunately, this approach is not yet viable because the adjoint, $\mathcal{L}_1^*$, is in general no longer surjective and we can no longer guarantee that $(\mathbf{u}, p, q) \in \mathcal{R}(\mathcal{L}_1^*)$, as the following example shows.

EXAMPLE 3.1. *Let Ω be the L-shaped domain from (2.15) and set $A = I$ and $\mathbf{b} = \mathbf{0}$. Let d be any positive analytic function and $c \in L^\infty$ with $0 < c < 1$ a.e. We choose homogeneous Dirichlet boundary conditions: $\Gamma_D = \partial\Omega$. Let Γ_H be the union of the three horizontal edges and Γ_V be the union of the three vertical edges of Ω. Thus, imposing $v_1 = 0$ on Γ_D is equivalent to imposing $\mathbf{n}\cdot(v_1, v_2) = 0$ on Γ_H and $\mathbf{t}\cdot(v_1, v_2) = 0$ on Γ_V, so*

$$\mathcal{H}_{DN}(\Omega) = H_H(\nabla\cdot;\Omega) \cap H_V(\nabla\times;\Omega)$$

holds for this example. The analysis of the div-curl operator in [11] shows that $\begin{bmatrix} -\nabla\cdot \\ \nabla\times \end{bmatrix}$ has a non-trivial null space on $\mathcal{H}_{DN}(\Omega)$. (For example, let $\mathbf{z} = \nabla\phi$, where $\Delta\phi = 0$, $\mathbf{n}\cdot\nabla\phi = 0$ on Γ_H, $\phi = 0$ on $\Gamma_{V_1} \cup \Gamma_{V_2}$ and $\phi = 1$ on Γ_{V_3}.) Let $\mathbf{z} \neq \mathbf{0}$ be such a null space element. Note that $\mathcal{L}_1(\mathbf{0}, \mathbf{z}) = (\mathbf{0}, -cz_1, -dz_2)^t$ is in $\left(L^2(\Omega)\right)^4$. Since $\mathcal{L}_1 = \mathcal{L}_1^*$ formally holds, Lemma 3.6 in [11] implies the existence of a more regular pre-image, $(\mathbf{v}, \mathbf{w}) \in \mathcal{D}(\mathcal{L}_1) \cap \left(H^1(\Omega)\right)^4$ with $\mathcal{L}_1(\mathbf{v}, \mathbf{w}) = (\mathbf{0}, -cz_1, -dz_2)^t$. Therefore, $(\mathbf{v}, \mathbf{w}-\mathbf{z})$ is a non-trivial element of null space $\mathcal{N}(\mathcal{L}_1)$. Since $(\mathcal{L}_1^*)^* = \mathcal{L}_1$, the closed range theorem implies that $\mathcal{R}(\mathcal{L}_1^*) = \mathcal{N}(\mathcal{L}_1)^\perp$, so $\mathcal{R}(\mathcal{L}_1^*)$ is not all of $\left(L^2(\Omega)\right)^4$ and $\mathcal{L}_1^*$ is not surjective.

To prove that, in general, $U = (\nabla p, p, 0) \notin \mathcal{R}(\mathcal{L}_1^*)$ so that the dual problem is not solvable, assume otherwise: $U \in \mathcal{R}(\mathcal{L}_1^*)$ or, equivalently, $U \perp \mathcal{N}(\mathcal{L}_1)$ for all admissible p. Let $(\mathbf{v}, \mathbf{w})$ be an element of $\mathcal{N}(\mathcal{L}_1)$, i.e.,

$$\mathbf{v} + \begin{bmatrix} -\nabla\cdot \\ \nabla\times \end{bmatrix} \mathbf{w} = \mathbf{0} \tag{3.3}$$

$$\begin{bmatrix} \nabla\cdot \\ -\nabla\times \end{bmatrix} \mathbf{v} - \begin{pmatrix} cw_1 \\ dw_2 \end{pmatrix} = \mathbf{0}. \tag{3.4}$$

Now, $U \perp (\mathbf{v}, \mathbf{w})$ means $\langle\nabla p, \mathbf{v}\rangle + \langle p, w_1\rangle = 0$. Using the divergence theorem and (3.4), we thus have $(c-1)\langle p, w_1\rangle = 0$. Since $\langle p, w_1\rangle$ must vanish for all admissible p, we must have $w_1 = 0$. From (3.3) and (3.4), we conclude that $w_2 \in H^1(\Omega)$ and

$$\Delta w_2 - dw_2 = 0.$$

The definition of $\mathcal{D}(\mathcal{L}_1)$ and equation (3.3) supply the boundary condition:

$$\mathbf{n}\cdot\nabla w_2 = -\mathbf{t}\cdot\nabla^\perp w_2 = -\mathbf{t}\cdot\mathbf{v} = 0 \text{ on } \Gamma_D = \partial\Omega.$$

Therefore, w_2 is a constant on Ω and (3.3)-(3.4) yield $\mathbf{v} = \mathbf{w} = \mathbf{0}$. Since $\mathcal{N}(\mathcal{L}_1)$ is nontrivial, our assumption is wrong, and the dual problem is not, in general, solvable in $\mathcal{D}(\mathcal{L}_1^*)$.

Our numerical experience supports the difficulty expressed in this example: it seems that $(\nabla p, p, 0)$ is in $\mathcal{R}(\mathcal{L}_1^*)$ only for very special choice of $A, \mathbf{b}, c, d$, and Ω, whenever $\partial\Omega$ contains irregular boundary points.

Nevertheless, the modifications that lead from $\mathcal{L}_0$ to $\mathcal{L}_1$ take a step in the right direction because we now have $\mathcal{D}(\mathcal{L}_1^*) \subset \left(H^1(\Omega)\right)^4$. This H^1-inclusion property guarantees that a dual solution, when it exists, can be easily approximated by standard H^1 finite element spaces. As the final step, we now modify the domain of the operators again to ensure solvability. The aim is to increase the domain of the new dual operator, $\mathcal{L}^*$, in order to make it surjective. This is done indirectly by reducing the number of boundary conditions on the dual domain. We do this by enforcing more boundary conditions on the domain of the primal operator, $\mathcal{L}$. Of course, we are only allowed to enforce additional boundary conditions on the primal problem that are fulfilled by the primal solution, $(\mathbf{u}, p, q)$. The key is to identify these allowable conditions and choose those that induce the appropriate $\mathcal{D}(\mathcal{L}^*)$.

First, we introduce the new modified operator, $\mathcal{L}$, then prove some useful lemmas, and finally we present our main results, the surjectivity of the dual operator, $\mathcal{L}^*$.

The two additional boundary conditions we enforce on the primal problem are

$$\int_{\Gamma_{N,i}} \mathbf{t} \cdot \mathbf{u} \, ds = 0, \quad i = 1, \ldots, M, \tag{3.5}$$

$$q = 0 \text{ on } \Gamma_Q \subset \Gamma_D. \tag{3.6}$$

These additional conditions are allowable because the primal solution, $(\mathbf{u}, p, q)$, satisfies them:

$$\int_{\Gamma_{N,i}} \mathbf{t} \cdot \mathbf{u} \, ds = \int_{\Gamma_{N,i}} \mathbf{t} \cdot \nabla p \, ds = \int_{\Gamma_{N,i}} \frac{dp}{ds} \, ds = 0, \quad i = 1, \ldots, M,$$

and $q = 0$ on $\partial\Omega$. For theoretical purposes, we impose condition (3.6) only on a subset, $\Gamma_Q \subset \Gamma_D$, that does not contain any irregular boundary points in its closure but has positive measure. See remark 3.2 below for motivation.

The new operator, $\mathcal{L}$, has the same form as $\mathcal{L}_1$, but differs again by its domain. We define the form of $\mathcal{L}$ blockwise:

$$T = \begin{bmatrix} \nabla\cdot A - \mathbf{b}\cdot \\ -\nabla\times \end{bmatrix}, \quad B = \begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix}, \tag{3.7}$$

$$S = \begin{bmatrix} -\partial_x & -\partial_y \\ -\partial_y & \partial_x \end{bmatrix} = \begin{bmatrix} -\nabla\cdot \\ \nabla\times \end{bmatrix} \tag{3.8}$$

$$\mathcal{L} = \begin{bmatrix} I & S \\ T & -B \end{bmatrix}. \tag{3.9}$$

The corresponding domains include the following additional boundary conditions:

$$\mathcal{D}(T) = \left\{ \mathbf{v} \in H_N(\nabla\cdot A; \Omega) \cap H_D(\nabla\times; \Omega) \ : \ \textstyle\int_{\Gamma_{N,i}} \mathbf{t} \cdot \mathbf{v} \, ds = 0, \ 1 \le i \le M \right\}, \tag{3.10}$$

$$\mathcal{D}(S) = \{(t, z) \in \mathcal{H}_{DN}(\Omega) \ : \ z = 0 \text{ on } \Gamma_Q\}, \tag{3.11}$$

$$\mathcal{D}(\mathcal{L}) = \mathcal{D}(T) \times \mathcal{D}(S). \tag{3.12}$$

These domains are Hilbert spaces under the div-curl norms,

$$||\mathbf{v}||_S^2 := ||\mathbf{v}||^2 + ||\nabla\cdot\mathbf{v}||^2 + ||\nabla\times\mathbf{v}||^2, \tag{3.13}$$

$$||\mathbf{v}||_T^2 := ||\mathbf{v}||^2 + ||\nabla\cdot(A\mathbf{v})||^2 + ||\nabla\times\mathbf{v}||^2, \tag{3.14}$$

$$||(\mathbf{v}, \mathbf{w})||_{\mathcal{L}}^2 := ||\mathbf{v}||_T^2 + ||\mathbf{w}||_S^2. \tag{3.15}$$

Integration by parts leads to the adjoint operators,

$$T^* = \begin{bmatrix} -(A\nabla + \mathbf{b}) & -\nabla^\perp \end{bmatrix}, \tag{3.16}$$

$$S^* = \begin{bmatrix} \partial_x & \partial_y \\ \partial_y & -\partial_x \end{bmatrix} = \begin{bmatrix} \nabla & \nabla^\perp \end{bmatrix}, \tag{3.17}$$

$$\mathcal{L}^* = \begin{bmatrix} I & T^* \\ S^* & -B \end{bmatrix}, \tag{3.18}$$

and the domains,

$$\mathcal{D}(S^*) = \left\{ \mathbf{v} \in \left(H^1(\Omega)\right)^2 \,:\, \mathbf{n}\cdot\mathbf{v} = 0 \text{ on } \Gamma_N \text{ and } \mathbf{t}\cdot\mathbf{v} = 0 \text{ on } \Gamma_D \setminus \Gamma_Q \right\}, \tag{3.19}$$

$$\mathcal{D}(T^*) = \left\{ (t,z) \in \left(H^1(\Omega)\right)^2 \,:\, t = 0 \text{ on } \Gamma_D \text{ and } z \equiv c_i \text{ on } \Gamma_{N,i} \right\}, \tag{3.20}$$

where $1 \le i \le M$, and c_i are arbitrary constants, and

$$\mathcal{D}(\mathcal{L}^*) = \mathcal{D}(S^*) \times \mathcal{D}(T^*). \tag{3.21}$$

REMARK 3.2. *We do not allow the closure of $\Gamma_Q \subset \Gamma_D$ to contain irregular boundary points because we would expect singular functions in $H(\nabla\times;\Omega) \cap H(\nabla\cdot;\Omega)$ to arise at these points. These singular functions would no longer be in $\mathcal{D}(S)$, but they would be in $\mathcal{D}(S^*)$. In practice, there seems to be no difficulty with allowing Γ_Q to touch irregular boundary points.*

For the remainder of this section, we adopt the assumptions on BVP (2.1)-(2.3) made in subsection 2.1 and let d be any nonnegative analytic function. We now prove coercivity of the primal operator, $\mathcal{L}$. To this end, we need two auxiliary results.

LEMMA 3.3. *$\mathcal{L}$ is injective.*

Proof: Assume that there exists a $(\mathbf{v},\mathbf{w}) \in \mathcal{D}(\mathcal{L})$ such that

$$\mathcal{L}(\mathbf{v},\mathbf{w}) = \begin{bmatrix} I & S \\ T & -B \end{bmatrix} \begin{pmatrix} \mathbf{v} \\ \mathbf{w} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix}.$$

Then $S\mathbf{w} \in \mathcal{D}(T)$, which, together with $\mathbf{w} \in \mathcal{D}(S)$, implies that

$$\mathbf{t}\cdot S\mathbf{w} = -\mathbf{n}\cdot\nabla w_2 = 0 \text{ on } \Gamma_Q, \tag{3.22}$$

$$\mathbf{n}\cdot AS\mathbf{w} = -\mathbf{n}\cdot A(\nabla w_1 + \nabla^\perp w_2) = 0 \text{ on } \Gamma_N. \tag{3.23}$$

Now choose any open set, $\mathcal{O} \subset \Omega$, such that $\overline{\mathcal{O}} \cap \Gamma_Q$ has positive measure and $\partial\mathcal{O}$ contains no irregular points of $\partial\Omega$. On $\mathcal{O}$ we have $\mathbf{w} \in (H^1)^2$. If we only look at the set $\mathcal{O}$, eliminating $\mathbf{v}$ yields the following equation for $\mathbf{w}$:

$$\begin{aligned} -TS\mathbf{w} - B\mathbf{w} &= -\begin{bmatrix} \nabla\cdot A - \mathbf{b}\cdot \\ -\nabla\times \end{bmatrix} \begin{bmatrix} -\nabla & -\nabla^\perp \end{bmatrix} \mathbf{w} - \begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix} \mathbf{w} \\ &= \begin{bmatrix} (\nabla\cdot A\nabla - \mathbf{b}\cdot\nabla - c) & (\nabla\cdot A\nabla^\perp - \mathbf{b}\cdot\nabla^\perp) \\ 0 & \Delta - d \end{bmatrix} \mathbf{w} = \mathbf{0}. \end{aligned} \tag{3.24}$$

Consider the second equation together with the boundary conditions to get

$$\begin{aligned} \Delta w_2 - d w_2 &= 0 \text{ in } \mathcal{O}, \\ w_2 &= 0 \text{ on } \partial\mathcal{O} \cap \Gamma_Q, \\ \mathbf{n}\cdot\nabla w_2 &= 0 \text{ on } \partial\mathcal{O} \cap \Gamma_Q. \end{aligned}$$

According to the unique continuation theorem (c.f. Holmgren and Hörmander [18]), we must have $w_2 = 0$ in $\mathcal{O}$. Since every point of Ω is in some $\mathcal{O}$ of this type, we conclude that $w_2 = 0$ in Ω. Since $S\mathbf{w} \in (L^2)^2$, then $w_2 = 0$ implies that $w_1 \in H^1(\Omega)$. Equations (3.23) and (3.24) now yield

$$\begin{aligned}(\nabla\cdot A\nabla - \mathbf{b}\cdot\nabla - c)w_1 &= 0 \text{ in } \Omega,\\ w_1 &= 0 \text{ on } \Gamma_D,\\ \mathbf{n}\cdot A\nabla w_1 &= 0 \text{ on } \Gamma_N.\end{aligned}$$

The well-posedness of BVP (2.1)-(2.3) implies that $w_1 = 0$ and, therefore, $\mathbf{v} = \mathbf{0}$. Hence, $\mathcal{N}(\mathcal{L}) = \{\mathbf{0}\}$ and the lemma follows.

□

LEMMA 3.4. *S is injective.*

Proof: Assume that there is a $\mathbf{w} \in \mathcal{D}(S)$ such that

$$S\mathbf{w} = \begin{bmatrix} -\nabla\cdot \\ \nabla\times \end{bmatrix}\mathbf{w} = \mathbf{0}.$$

Since Ω is simply connected, the curl-free condition here implies that $\mathbf{w} = \nabla\phi$ for some $\phi \in H^1(\Omega)$, with ϕ determined uniquely up to a constant (c.f.[16]). The div-free condition implies that ϕ is harmonic. The boundary conditions on $\mathcal{D}(S)$ imply that $\nabla\phi = \mathbf{0}$ on Γ_Q, so

$$\mathbf{n}\cdot\nabla\phi = 0 \text{ on } \Gamma_Q.$$

But $\mathbf{t}\cdot\nabla\phi = 0$ is also true on Γ_Q. Thus, ϕ is constant on Γ_Q and, without loss of generality, we may assume

$$\phi = 0 \text{ on } \Gamma_Q.$$

Applying the unique continuation theorem (c.f. Hörmander and Holmgren [18]) yields $\phi = 0$, which completes the proof.

□

We are now able to establish coercivity of $\mathcal{L}$.

THEOREM 3.5. *Operators S, T, and $\mathcal{L}$ are coercive in the norms (3.13), (3.14) and (3.15), respectively.*

Proof: We begin by proving coercivity of S and T. For S, it suffices to prove a Poincaré inequality of the form:

There exists constant $C > 0$ such that

$$\|\mathbf{w}\|^2 \le C\left(\|\nabla\cdot\mathbf{w}\|^2 + \|\nabla\times\mathbf{w}\|^2\right) \tag{3.25}$$

for all $\mathbf{w} \in \mathcal{D}(S)$.

To establish (3.25), we assume that no such inequality exists, that is, that there exists $\{\mathbf{w}^{(i)}\}_{i=1,\infty} \in \mathcal{D}(S)$ such that, for all $i > 0$,

$$||\nabla\cdot\mathbf{w}^{(i)}||^2 + ||\nabla\times\mathbf{w}^{(i)}||^2 = 1, \tag{3.26}$$

$$||\mathbf{w}^{(i)}||^2 \ge i. \tag{3.27}$$

Now, every $\mathbf{w}^{(i)} \in \mathcal{D}(S)$ can be written as

$$\mathbf{w}^{(i)} = \mathbf{z}^{(i)} + \sum_{j=1}^{K}\beta_{ij}\boldsymbol{\phi}^{(j)},$$

where $\mathbf{z}^{(i)} \in \mathcal{D}(S) \cap \left(H^1(\Omega)\right)^2$ and $\{\boldsymbol{\phi}^{(j)}\}_{j=1,K}$ is a basis of the finite-dimensional orthogonal complement of $\mathcal{D}(S) \cap \left(H^1(\Omega)\right)^2$ in $\mathcal{D}(S)$. Here, we take orthogonality in the $\mathcal{H}_{DN}(\Omega)$ sense, which is an inner product because S is injective. That is, we require

$$\left\langle \nabla\cdot\boldsymbol{\phi}^{(j)}, \nabla\cdot\mathbf{z} \right\rangle + \left\langle \nabla\times\boldsymbol{\phi}^{(j)}, \nabla\times\mathbf{z} \right\rangle = 0$$

for every $\mathbf{z} \in \mathcal{D}(S) \cap \left(H^1(\Omega)\right)^2$. Then, (3.26) becomes

$$||\nabla\cdot\mathbf{z}^{(i)}||^2 + ||\nabla\times\mathbf{z}^{(i)}||^2 + ||\nabla\cdot\sum_{j=1}^{K}\beta_{ij}\boldsymbol{\phi}^{(j)}||^2 + ||\nabla\times\sum_{j=1}^{K}\beta_{ij}\boldsymbol{\phi}^{(j)}||^2 = 1. \tag{3.28}$$

Since $\mathbf{z}^{(i)} \in \left(H^1(\Omega)\right)^2$, we know that there exist constants, $C_0, C_1 > 0$, such that, for all $i > 0$,

$$||\mathbf{z}^{(i)}||^2 \leq C_0\left(||\nabla\mathbf{z}^{(i)}||^2\right) \leq C_1\left(||\nabla\cdot\mathbf{z}^{(i)}||^2 + ||\nabla\times\mathbf{z}^{(i)}||^2\right) \leq C_1, \tag{3.29}$$

where the second inequality can be found in [17] and the last inequality follows from (3.28). In several places in this proof, we make use of the general inequality

$$||\alpha+\beta||^2 \leq 2(||\alpha||^2 + ||\beta||^2). \tag{3.30}$$

Now, to satisfy (3.27), we combine it with (3.28) and (3.29), using inequality (3.30), to see that we must have

$$||\sum_{j=1}^{K}\beta_{ij}\boldsymbol{\phi}^{(j)}||^2 \geq \frac{i}{2} - C_1 \tag{3.31}$$

for all $i > 0$. We now define $P, N \in \mathbb{R}^{(K\times K)}$ as

$$\begin{aligned} P &:= (p_{kl}) = \left\langle \boldsymbol{\phi}^{(k)}, \boldsymbol{\phi}^{(l)} \right\rangle \\ N &:= (n_{kl}) = \left\langle \nabla\cdot\boldsymbol{\phi}^{(k)}, \nabla\cdot\boldsymbol{\phi}^{(l)} \right\rangle + \left\langle \nabla\times\boldsymbol{\phi}^{(k)}, \nabla\times\boldsymbol{\phi}^{(l)} \right\rangle. \end{aligned}$$

Because the $\{\boldsymbol{\phi}^{(j)}\}_{j=1,K}$ are linearly independent and S has no null space (see Lemma 3.4), P and N must be symmetric positive definite matrices. Now, define the vectors

$$\mathbf{b}^{(i)} := (\beta_{i1}, \beta_{i2}, \ldots, \beta_{iK})^t.$$

Equations (3.28) and (3.31) imply

$$\frac{\mathbf{b}^{(i)}\cdot P\mathbf{b}^{(i)}}{\mathbf{b}^{(i)}\cdot N\mathbf{b}^{(i)}} \geq \frac{i}{2} - C_1,$$

which contradicts positive definiteness of N. Therefore, (3.25) holds and S is coercive in the norm defined by (3.13).

To prove coercivity of T, note that, by inequality (3.30), there exists constant $C_2 > 0$ (dependent only on $||\mathbf{b}||$) such that

$$\begin{aligned} ||\mathbf{v}||_T^2 &\leq ||\mathbf{v}||^2 + 2||\nabla\cdot(A\mathbf{v}) - \mathbf{b}\cdot\mathbf{v}||^2 + ||\nabla\times\mathbf{v}||^2 + 2||\mathbf{b}\cdot\mathbf{v}||^2 \\ &\leq C_2\left(||T\mathbf{v}||^2 + ||\mathbf{v}||^2\right) \end{aligned}$$

for all $\mathbf{v} \in \mathcal{D}(T)$. Since T is injective (c.f. [11]) and $\mathcal{D}(T)$ is compactly embedded in $\left(L^2(\Omega)\right)^2$, a standard compactness argument establishes coercivity of T.

By coercivity of T and S and the inequality (3.30), there exist constants $C_3, C_4 > 0$ (depending only on $||\mathbf{b}||$, $||c||$, and $||d||$) such that, for all $(\mathbf{v}, \mathbf{w}) \in \mathcal{D}(\mathcal{L})$,

$$\begin{aligned} ||(\mathbf{v}, \mathbf{w})||^2_{\mathcal{L}} = ||\mathbf{v}||^2_T + ||\mathbf{w}||^2_S &\leq C_3 \left(||T\mathbf{v}||^2 + ||S\mathbf{w}||^2 \right) \\ &\leq C_4 \left(||T\mathbf{v} - B\mathbf{w}||^2 + ||\mathbf{v} + S\mathbf{w}||^2 + ||\mathbf{v}||^2 + ||\mathbf{w}||^2 \right). \end{aligned}$$

Since $\mathcal{L}$ is injective (see lemma 3.3) and $\mathcal{D}(\mathcal{L})$ is compactly embedded in $\left(L^2(\Omega)\right)^4$, again we appeal to a standard compactness argument and establish coercivity of $\mathcal{L}$.

□

The main result of this section is the following theorem, which confirms the existence of a dual solution. It is a simple consequence of theorem 3.5.

THEOREM 3.6. *$\mathcal{L}^* : \mathcal{D}(\mathcal{L}^*) \to \left(L^2(\Omega)\right)^4$ is surjective.*

Proof: It is clear that $(\mathcal{L}^*)^* = \mathcal{L}$. Thus, both $\mathcal{L}$ and $\mathcal{L}^*$ are closed and we may use the closed range theorem. Since $\mathcal{L}$ is coercive (see Theorem 3.5), then $\mathcal{R}(\mathcal{L})$ is closed. The closed range theorem implies that $\mathcal{R}(\mathcal{L}^*)$ is closed. Thus, we know that $\mathcal{R}(\mathcal{L}^*) = \mathcal{N}(\mathcal{L})^{\perp}$. Since $\mathcal{N}(\mathcal{L})$ is empty, then $\mathcal{L}^*$ is surjective.

□

3.1. The case $c = 0$. This is the case that was examined in [12]. In this paper we remove the requirement that Γ_D and Γ_N have at most one component. When $c = 0$, it is useful to choose $d = 0$ because the second and third rows of $\mathcal{L}_0$ in (3.1) only involve $\mathbf{u}$. This allows us to write the primal problem, $\mathcal{L}U = F$, in two-stages:

$$T\mathbf{u} = \begin{pmatrix} f \\ 0 \end{pmatrix}, \tag{3.32}$$

$$\nabla p = \mathbf{u}. \tag{3.33}$$

Since T is injective by itself (see the proof of theorem 3.5), problem (3.32) alone is sufficient to determine $\mathbf{u}$. We can, thus, begin by solving this so-called first stage equation. The second stage, equation (3.33), can be then be solved for p if needed.

Discontinuous coefficients in A and irregular boundary points on $\partial\Omega$ imply $\mathbf{u} \notin \left(H^1(\Omega)\right)^2$, so we use a FOSLL* formulation to solve (3.32). To accommodate the possibility of multiply connected boundary components, (3.32) is posed on domain $\mathcal{D}(T)$ defined in (3.10). Note, then, that the dual problem for the first stage reads $T^*\mathbf{w} = \mathbf{u}$ and takes the variational form

$$\langle T^*\mathbf{w}, T^*\mathbf{v} \rangle = \left\langle (f, 0)^t, \mathbf{v} \right\rangle, \text{ for all } \mathbf{v} \in \mathcal{D}(T^*). \tag{3.34}$$

THEOREM 3.7. *Operator $T^* : \mathcal{D}(T^*) \to \left(L^2(\Omega)\right)^2$ is surjective.*

Proof: Coercivity of T was proved for Theorem 3.5 and, by arguments similar to those in Theorem 3.6, we can then prove surjectivity of T^*.

□

This theorem establishes existence of a solution for the first stage (3.32). Note that recovery of the solution, p, of the original BVP, (2.1)-(2.3), can then be done by applying a standard FOSLS scheme for the second stage equation, (3.33), i.e., by minimizing $||\nabla p - \mathbf{u}||$, where $\mathbf{u} = T^*\mathbf{w}$ is the approximation obtained in the first stage. This minimization is done in an appropriate subspace of $H^1_D(\Omega)$ and leads to an H^1 approximation of p, which is clearly more desirable than the L^2 approximations for p obtained by the general FOSLL* approach for $c \neq 0$.

REMARK 3.8. *A closer look at the dual problem for the first stage, $T^*\mathbf{w} = \mathbf{u}$, shows that the second component of the dual variable, w_2, is only determined up to a constant. Therefore, without*

loss of generality, we can restrict the space in which we are looking for $\mathbf{w}$ *to*

$$\Big\{\mathbf{w} \in \big(H^1(\Omega)\big)^2 \ : \ w_1 = 0 \ on \ \Gamma_D, \\ w_2 = 0 \ on \ \Gamma_{N,1}, w_2 \equiv c_i \ on \ \Gamma_{N,i}, \ 2 \le i \le M\Big\}.$$

Thus, for the case $c = 0$*, standard FOSLL** *, as proposed in [12], works well enough, unless* Γ_D *or* Γ_N *is not simply connected.*

REMARK 3.9. *A scaled version of the first stage (3.32) that solves for the scaled flux,* $\widetilde{\mathbf{u}} = A^{1/2}\nabla p$ *(see (3.36)), yields a dual problem with better computational performance when used in conjunction with multigrid solvers.*

3.2. The case $\mathbf{b} = \mathbf{0}$. For $\mathbf{b} = \mathbf{0}$, we consider two cases: $c > 0$ and $c = 0$. (We exclude the case that c is neither 0 nor strictly positive.) We show in both cases that a scaled form of FOSLL* reduces to the standard Galerkin formulation of (2.1)-(2.3).

Consider first the case $c > 0$. We can rescale the primal problem by using the scaled primal unknowns, $(\tilde{\mathbf{u}}, \tilde{p}, \tilde{q}) = (A^{1/2}\nabla p, c^{1/2}p, c^{1/2}q)$. The primal operator is then simply a scaled version of L_1 used in standard FOSLL* . The primal problem takes the form

$$\begin{aligned} \tilde{L}_1(\tilde{\mathbf{u}}, \tilde{p}, \tilde{q}) &= (\mathbf{0}, f, 0)^t \text{ in } \Omega, \\ \mathbf{t} \cdot A^{-1/2}\tilde{\mathbf{u}} &= 0 \text{ on } \Gamma_D, \\ \mathbf{n} \cdot A^{1/2}\tilde{\mathbf{u}} &= 0 \text{ on } \Gamma_N, \\ \tilde{p} &= 0 \text{ on } \Gamma_D, \\ \tilde{q} &= 0 \text{ on } \Gamma_N, \end{aligned}$$

where

$$\begin{aligned} \tilde{L}_1(\tilde{\mathbf{u}}, \tilde{p}, \tilde{q}) &:= \begin{bmatrix} A^{-1/2} & -\nabla c^{-1/2} & -\nabla^\perp c^{-1/2} \\ \nabla\cdot A^{1/2} & -c^{1/2} & 0 \\ -\nabla\times A^{-1/2} & 0 & 0 \end{bmatrix} \begin{pmatrix} \tilde{\mathbf{u}} \\ \tilde{p} \\ \tilde{q} \end{pmatrix} \\ &= \begin{pmatrix} A^{-1/2}\tilde{\mathbf{u}} - \nabla c^{-1/2}\tilde{p} - \nabla^\perp c^{-1/2}\tilde{q} \\ \nabla\cdot(A^{1/2}\tilde{\mathbf{u}}) - c^{1/2}\tilde{p} \\ -\nabla\times(A^{-1/2}\tilde{\mathbf{u}}) \end{pmatrix}. \end{aligned}$$

The domain of this operator is

$$\mathcal{D}(\tilde{L}_1) = \Big(H_N(\nabla\cdot A^{1/2}; \Omega) \cap H_D(\nabla\times A^{-1/2}; \Omega)\Big) \times H^1_D(c^{-1/2}; \Omega) \times H^1_N(c^{-1/2}; \Omega),$$

where $\phi \in H^1_J(c^{-1/2}; \Omega)$ if and only if $c^{-1/2}\phi \in H^1_J(\Omega)$. Thus, the dual problem takes the form

$$\tilde{L}_1^*(\mathbf{w}, r, s) = (\tilde{\mathbf{u}}, \tilde{p}, \tilde{q})^t = (A^{1/2}\nabla p, c^{1/2}p, 0)^t \text{ in } \Omega \tag{3.35}$$

on

$$\mathcal{D}(\tilde{L}_1^*) = (H_N(\nabla\cdot; \Omega) \cap H_D(\nabla\times; \Omega)) \times H^1_D(\Omega) \times H^1_N(\Omega),$$

where the adjoint operator has the form

$$\tilde{L}_1^*(\mathbf{w}, r, s) = \begin{bmatrix} A^{-1/2} & -A^{1/2}\nabla & -A^{-1/2}\nabla^\perp \\ c^{-1/2}\nabla\cdot & -c^{1/2} & 0 \\ -c^{-1/2}\nabla\times & 0 & 0 \end{bmatrix} \begin{pmatrix} \mathbf{w} \\ r \\ s \end{pmatrix}.$$

Remarkably, this specially scaled problem has a dual solution in $\left(H^1(\Omega)\right)^4$, namely,

$$(\mathbf{w}, r, s) = (\mathbf{0}, -p, 0).$$

Knowing that $\mathbf{w}$ and s vanish in this special case, dual problem (3.35) takes the following variational form:

Find $p \in H^1_D(\Omega)$ such that

$$\langle A\nabla p, \nabla t\rangle + \langle cp, t\rangle = -\langle f, t\rangle$$

for all $t \in H^1_D(\Omega)$.

This is precisely the variational form of the Galerkin approach for BVP (2.1)-(2.3) with $\mathbf{b} = \mathbf{0}$. In other words, FOSLL* yields the same H^1 approximation, p^h, as the Galerkin approach.

Next, consider the case $\mathbf{b} = \mathbf{0}$ and $c = 0$. A scaled two-stage approach leads to the following first stage primal problem:

$$\begin{bmatrix} \nabla\cdot A^{1/2} \\ -\nabla\times A^{-1/2} \end{bmatrix} \tilde{\mathbf{u}} = \begin{pmatrix} f \\ 0 \end{pmatrix} \tag{3.36}$$

on $H_N(\nabla\cdot A^{1/2}; \Omega) \cap H_D(\nabla\times A^{-1/2}; \Omega)$. The corresponding dual problem is

$$-A^{1/2}\nabla w_1 - A^{-1/2}\nabla^\perp w_2 = \tilde{\mathbf{u}} = A^{1/2}\nabla p$$

on $H^1_D(\Omega) \times H^1_N(\Omega)$, which obviously has the solution $\mathbf{w} = (-p, 0)$. Knowing that w_2 vanishes leads to the following variational form:

Find $p \in H^1_D(\Omega)$ such that

$$\langle A\nabla p, \nabla t\rangle = -\langle f, t\rangle$$

for all $t \in H^1_D(\Omega)$.

This, again, is precisely the variational form of the Galerkin approach for BVP (2.1)-(2.3) when $\mathbf{b} = \mathbf{0}$ and $c = 0$. Thus, FOSLL* and Galerkin again yield the same H^1 approximation, p^h.

4. Numerical Results. Here we report on various numerical results and discuss some implementation issues for the methods proposed in the previous section. All problems in this section were computed with FOSPACK [22]. The linear solver used for the discretized equations was a conjugate gradient iteration (PCG), preconditioned by algebraic multigrid (AMG) using one standard W(1,1)-cycle based on point Gauss-Seidel relaxation. In all cases, the PCG/AMG iterations were applied until the residual norm of the linear system was reduced by a factor of at least 10^{-10}. While this criterion is unnecessarily strong, and is not recommended in practice, it was used to eliminate algebraic error from the analysis of the convergence of the finite element approximations.

First, we show how the improved FOSLL* method performs on the problem proposed in Examples 2.1 and 2.3. One of our improvements was the introduction of $\Gamma_Q \subset \Gamma_D$, an additional Dirichlet boundary for the slack variable. For our numerical tests, we chose the domain described in Example 2.1 and

$$\Gamma_Q = \{(x, y) \in \Gamma_D : x \in (0.5, 1) \text{ and } y = 0\}. \tag{4.1}$$

EXAMPLE 4.1. *We apply the improved FOSLL* method to the BVP from Examples 2.1 and 2.3. We, thus, use the constructs defined in (3.7)-(3.21), with Γ_Q as in (4.1) and $d = 1$. The L^2 norms of the errors are shown in Table 4.1. Since the primal solution is in $(H^\alpha(\Omega))^4$ only for $\alpha < 2/3$, the optimal asymptotic bounds on these errors is in general proportional to $h^{2/3}$. We compute the*

h	1/4	1/8	1/16	1/32	1/64	1/128	1/256
$\|\|p-p^h\|\|_0$	.0475	.0189	.0113	.0075	.0047	.0027	.0016
β		1.328	.738	.606	.674	.767	.809
$\|\|\mathbf{u}-\mathbf{u}^h\|\|_0$	.6674	.3051	.1573	.0810	.0421	.0222	.0120
β		1.129	.956	.958	.946	.923	.892
ρ	0.20	0.31	0.46	0.65	0.77	0.83	0.87

TABLE 4.1

Error norms, approximate order of discretization convergence, β, and asymptotic AMG convergence factors, ρ, for the improved FOSLL* approximations for Example 4.1 on a sequence of uniform meshes with meshsizes h.

approximate order of convergence by computing β such that $(1/2)^\beta$ is equal to the ratio of errors on consecutive grids. The table suggests that the improved FOSLL* approach does indeed achieve these optimal bounds, while the FOSLS and standard FOSLL* methods do not converge at all (cf. Tables 2.1 and 2.2).

For the improved FOSLL* method, the four components of the dual solution, $(\mathbf{w}^h, r^h, s^h)$, on the $h = 1/32$ mesh are shown in Figure 4.1. By simply computing $(\mathbf{u}^h, p^h, q^h)^t = \mathcal{L}^*(\mathbf{w}^h, r^h, s^h)$, we obtained L^2 approximations for the primal variables, as shown in Figures 4.2. As these figures

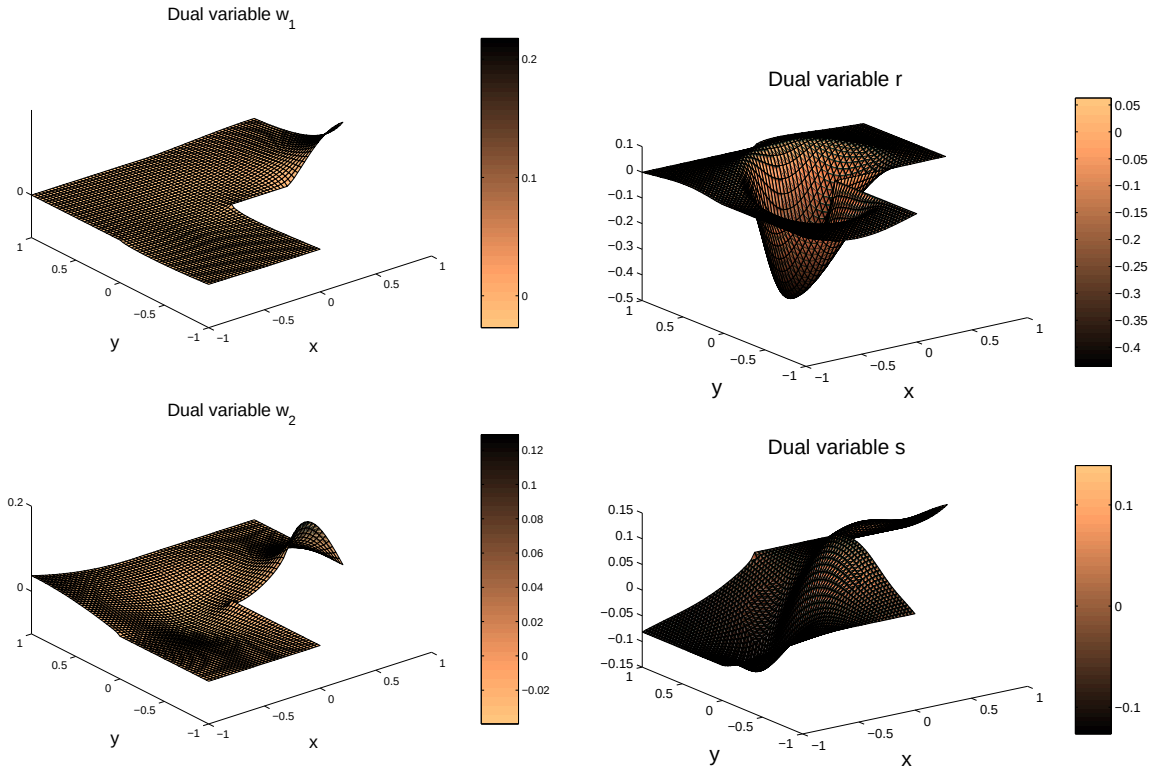

FIG. 4.1. *Approximations of the dual variables for Example 4.1 on a uniform mesh with $h = 1/32$.*

and tables show, the improved FOSLL* method yields converging L^2 approximations for the primal variables, $\mathbf{u}$ and p.

Unfortunately, convergence of $\|p - p^h\|_0$ tends to drop to a suboptimal rate if Γ_Q is chosen to be too small, especially when there are irregular points inside the Neumann boundary. Therefore, one has to take care in choosing Γ_Q sufficiently large. On the other hand, choosing such a large

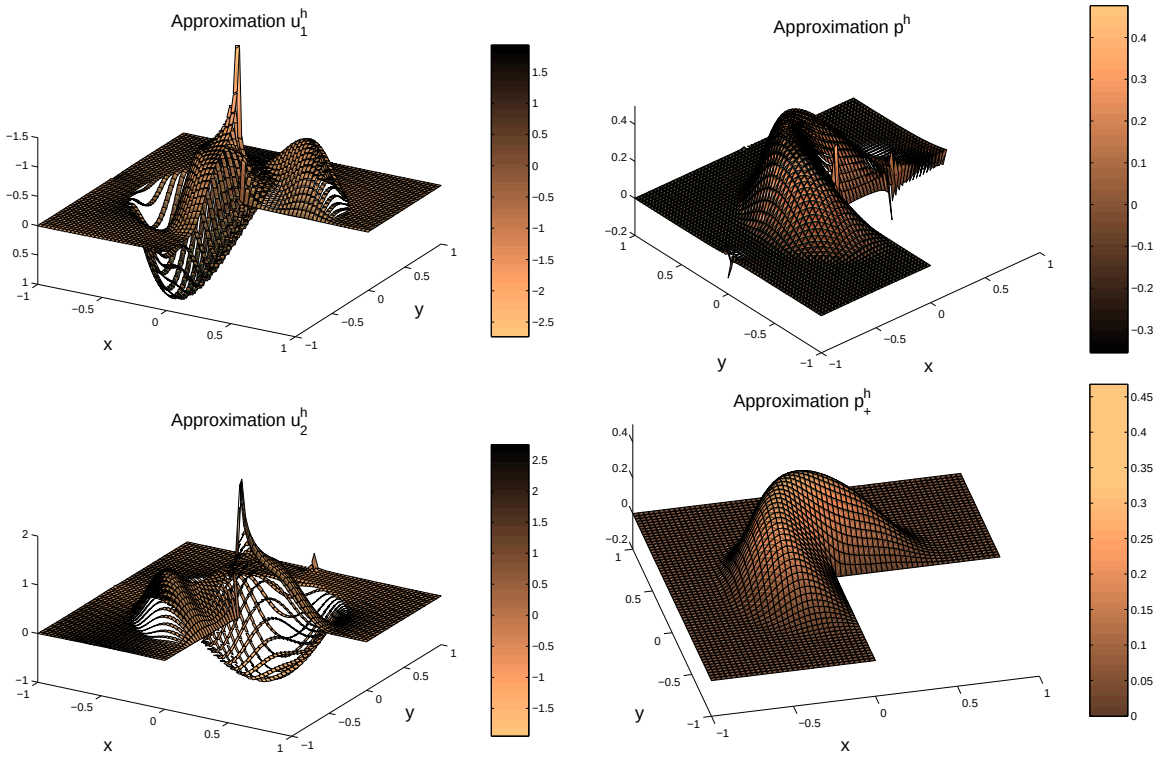

FIG. 4.2. *Approximations p^h and $\mathbf{u}^h$ of primal variables p and $\mathbf{u} = \nabla p$ as well as second-stage approximation p^h_+ of p on a uniform mesh with $h = 1/32$ for Example 4.1.*

Γ_Q with a fixed length seems to inhibit optimal AMG performance: the average per-step residual error reduction factor, which we call ρ, seems to depend on the meshsize, h. In fact, $1-\rho$ seems to be proportional to h^α for some positive α. For Example 4.1, the multigrid reduction factor given in Table 4.1 suggests that $1-\rho$ is proportional to $h^{3/4}$. This difficulty seems to come from the null space of $\mathcal{L}^*$ as defined in (3.7)-(3.21). This null space is non-trivial since there are no boundary conditions for the first two components of $\mathcal{D}(\mathcal{L}^*)$ on Γ_Q. One remedy could be to use linear solvers that can deal with non-trivial null spaces, such as the MINRES algorithm (cf. [15]). Another remedy is discussed in the next subsection.

REMARK 4.2. *For $c = 0$, the primal problem can be decomposed as shown in (3.32)-(3.33). Problem (3.32) can be solved by a FOSLL* method. Since this problem only involves the operator T, no Γ_Q is needed. Furthermore, the dual problem of this first stage has a full set of boundary conditions, namely, $w_1 = 0$ on Γ_D, $w_2 = 0$ on $\Gamma_{N,1}$, and $w_2 \equiv const$ on $\Gamma_{N,i}$, $2 \le i \le M$. This full set of boundary conditions leads to optimal multigrid convergence for $c = 0$. To demonstrate this, we tested this approach on Example 4.1 with $c = 0$. Table 4.2 shows the approximation errors and the multigrid convergence factors for both stages. Here, second-stage equation (3.33) is solved by FOSLS , since we know $p \in H^1(\Omega)$ and can therefore obtain H^1 approximations for p. Both FE and multigrid convergence show optimal behavior.*

4.1. Restoring optimal multigrid convergence. A heuristic approach for restoring optimal multigrid convergence (i.e., $\rho \ll 1$) is to choose different boundaries Γ^h_Q on different meshes so that $|\Gamma^h_Q| = O(h)$. The motivation for this is that such a choice for Γ^h_Q should control the dimension of the null space of the discrete operator since only a bounded number of elements could then intersect Γ^h_Q. These null space components that AMG cannot seem to eliminate by itself would then hopefully

h	1/4	1/8	1/16	1/32	1/64	1/128	1/256
$\|\|\mathbf{u}-\mathbf{u}^h\|\|_0$	.4588	.2036	.1046	.0539	.0281	.0149	.0081
β		1.1718	.961	.956	.941	.916	.882
ρ	0.12	0.10	0.054	0.041	0.032	0.040	0.040
$\|\|p-p^h\|\|_0$	4.41E-2	1.21E-2	3.34E-3	1.02E-3	3.34E-4	1.19E-4	4.63E-5
β		1.856	1.868	1.710	1.614	1.482	1.367
ρ	0.022	0.032	0.032	0.031	0.040	0.031	0.031

TABLE 4.2

Error norms and multigrid convergence for the approximations for Example 4.1 for $c = 0$ on a sequence of uniform meshes with meshsizes h. Upper half: First stage, using the improved FOSLL* method. Lower half: Second stage, using FOSLS method.

be attenuated by a fixed number of conjugate gradient steps.

The new difficulty that this choice introduces is that operators S^* and $\mathcal{L}^*$ lose surjectivity in the limit $h \to 0$. This, in turn, impairs finite element convergence as h decreases. Fortunately, this difficulty does not effect coercivity of T nor, as our observations show, convergence of $\|\mathbf{u}^h - \mathbf{u}\|$. Convergence of $||p^h - p||$ does degrade, however, but this can be remedied by appealing to the relation $\nabla p = \mathbf{u}$. That is, we can simply replace p^h by a new approximation, denoted p^h_+, that approximately solves $\nabla p^h_+ = \mathbf{u}^h$, where $\mathbf{u}^h$ is the approximation for $\mathbf{u}$ obtained by the improved FOSLL* method with variable Γ^h_Q. This post processing step is exactly the same as in (3.33), so we refer to it as the second stage. Since convergence of $\|\mathbf{u}^h - \mathbf{u}\|$ is still optimal for Γ^h_Q, then convergence of $\|p^h_+ - p\|$ should be optimal as well. Our implementation solves this second stage by the FOSLS approach of finding $p^h_+ = \arg\min \|\nabla z - \mathbf{u}^h\|_0$ where z is chosen from the same H^1-conforming finite element space that was used to approximate the dual solution.

EXAMPLE 4.3. *Using the same problem as in Example 4.1, we make a different choice for Γ_Q:*

$$\Gamma^h_Q = \{(x, y) \in \Gamma_D : x \in (1 - 4h, 1), y = 0\},$$

for which $|\Gamma^h_Q| = 4h$. In Table 4.3, we list the L^2 errors associated with $\mathbf{u}^h$, p^h, and p^h_+. We also include the multigrid convergence factors, ρ, for the solution of the discretized dual problem and the computational cost of the second stage as a percentage of the computational cost of the solution of the discretized dual problem. The results show that our approach leads to optimal multigrid convergence and a very accurate approximation for p at very small additional cost. Approximations p^h_+ and p^h for this problem on a mesh with $h = 1/32$ are shown in Figure 4.2.

REMARK 4.4. *The second stage yields an H^1 approximation to p, while p^h is in general in $L^2 \setminus H^1$. This desirable feature of this new approach utilizes the higher regularity of p in an efficient way. The approximation is not only in a smoother space, but also more accurate. Thus, the second stage is generally an effective tool to improve convergence of FOSLL* , not only just for the case of variable Γ^h_Q.*

4.2. Dependence of A and $\mathbf{b}$. Here we report on examples that demonstrate how FOSLL* depends on A and $\mathbf{b}$.

EXAMPLE 4.5. *In a first experiment, we slightly changed Example 4.3 by setting $A = \sigma I$, with $\sigma = 1$ for $x + y < 0$ and $\sigma = \sigma_0$ otherwise. The results are displayed in Table 4.4 and show that the AMG solver works well, even in the presence of huge jumps in the coefficients. It is remarkable that the AMG convergence factors are getting better for very fine meshes, where $1/h$ starts to dominate the convection and the jumping coefficients. For this example, we used the scaling mentioned in Remark 3.9.*

As a second experiment, we fixed $\sigma_0 = 1$ and varied the convection, $\mathbf{b}$. The results of this experiment are shown in Table 4.5. Note again the relative insensitivity of the order of discretization error, now with respect to the size of $\mathbf{b}$. AMG performance does degrade with increasing size of $\mathbf{b}$,

h	1/4	1/8	1/16	1/32	1/64	1/128	1/256
$\|\|p-p^h\|\|_0$	.0475	.0194	.0125	.0093	.0071	.0055	.0043
β		1.29	.639	.430	.385	.372	365
$\|\|\mathbf{u}-\mathbf{u}^h\|\|_0$	.6674	.3051	.1573	.0810	.0420	.0221	.0120
β		1.13	.956	.958	.946	.924	.892
ρ	0.19	0.23	0.17	0.13	0.10	0.08	0.09
$\|\|p-p_+^h\|\|_0$	4.35E-2	1.20E-2	3.36E-3	1.09E-3	3.95E-4	1.64E-4	7.56E-5
β		1.857	1.837	1.625	1.464	1.271	1.113
$\|\|\mathbf{u}-\nabla p_+^h\|\|_0$	.4640	.2078	.1071	.0555	.0290	.0155	.0084
β		1.159	.955	.950	.934	.908	.874
ρ	0.022	0.032	0.032	0.031	0.040	0.031	0.031
$stage2$	3.1%	3.4%	3.7%	4.0%	5.0%	4.2%	4.5 %

TABLE 4.3

Upper half: Error norms, approximate order of discretization convergence, β, and multigrid convergence factors, ρ, for the improved FOSLL* approximations for Example 4.1 on a sequence of uniform meshes with meshsizes h and $\Gamma_Q^h \in O(h)$. Lower half: Error norms for the second-stage approximation, approximate order of discretization convergence, β, and work of the second stage as a percentage of the work of the FOSLL* method above.

but this reflects the usual behavior of standard *multigrid solvers for convection dominated problems. Again, as the meshsize tends to* 0*, the discretized differential operators dominate the convection and cause a steady improvement of the AMG convergence rates.*

		h						
σ_0		1/4	1/8	1/16	1/32	1/64	1/128	1/256
10^0	$\|\|\mathbf{u}-\mathbf{u}^h\|\|_0$	.6674	.3051	.1573	.0810	.0420	.0221	.0120
	β		1.13	.956	.958	.946	.924	.892
	ρ	0.19	0.23	0.17	0.13	0.10	0.08	0.09
10^3	$\|\|\mathbf{u}-\mathbf{u}^h\|\|_0$	.6732	.3277	.1672	.0882	.0483	.0276	.0166
	β		1.04	.971	.923	.869	.805	.734
	ρ	0.38	0.47	0.53	0.49	0.40	0.31	0.23
10^6	$\|\|\mathbf{u}-\mathbf{u}^h\|\|_0$	.6738	.3280	.1674	.0883	.0484	.0277	.0166
	β		1.04	.971	.923	.869	.805	.734
	ρ	0.22	0.24	0.37	0.47	0.62	0.76	0.77

TABLE 4.4

Error norms and AMG convergence factors for the approximations from Example 4.5 for varying σ_0 on a sequence of uniform meshes with meshsizes h.

5. Conclusions. In this paper we have developed new FOSLL* formulations that allow the use of H^1-conforming finite element spaces and optimal multigrid solvers for constructing L^2 approximations of the primal variables on an extended class of scalar elliptic equations. This class includes problems with reaction terms, domains with Dirichlet and Neumann boundaries with multiple components, and irregular boundary points. The extension was accomplished by redefining the boundary conditions associated with the slack variables in the primal problem. Specifically, for domains with $\Gamma_D \neq \emptyset$, the slack variable, q, was given additional boundary conditions on $\Gamma_Q \subset \Gamma_D$. Our theory establishes the surjectivity of the adjoint operator, $\mathcal{L}^*$, as long as Γ_Q containes no irregular points. However, numerical results show that the multilevel solution techniques work better, and the finite element approximations are no worse, if Γ_Q is chosen to touch irregular boundary points and to shrink along with the mesh spacing, h. The case of pure Neumann boundary conditions remains an open problem.

The improved FOSLL* approach yields an L^2 approximation to the primal flux variable that

$\mathbf{b}^t$		h 1/16	1/32	1/64	1/128	1/256
$(\frac{-y}{10}, 10x)$	$\|\|\mathbf{u}-\mathbf{u}^h\|\|_0$	.1573	.0810	.0420	.0221	.0119
	β		.958	.946	.924	.892
	ρ	0.17	0.13	0.10	0.08	0.09
$10(\frac{-y}{10}, 10x)$	$\|\|\mathbf{u}-\mathbf{u}^h\|\|_0$	.3356	.1862	.0968	.0492	.0250
	β		0.850	0.944	0.978	0.979
	ρ	0.57	0.60	0.58	0.45	0.31
$100(\frac{-y}{10}, 10x)$	$\|\|\mathbf{u}-\mathbf{u}^h\|\|_0$	.4687	.3371	.2230	.1342	.0737
	β		0.476	0.596	0.732	0.864
	ρ	0.65	0.76	0.83	0.84	0.84

TABLE 4.5

Error norms, approximate order of discretization convergence, β, and AMG convergence factors for the approximations from Example 4.5 for varying $\mathbf{b}$ *on a sequence of uniform meshes with meshsizes* h.

achieves the optimal theoretical convergence rate. A post processing step was shown to yield optimal H^1 approximation to the original scalar variable, p, at a small additional cost.

We also showed that the FOSLL* formulation produces the same approximation as a Galerkin formulation of the original second-order boundary value problem, (2.1)-(2.3), in the absence of first order terms ($\mathbf{b} = \mathbf{0}$) and either no reaction term ($c = 0$) or strictly positive reaction term ($c > 0$).

The efficiency of the improved FOSLL* formulations was illustrated by a series of numerical examples.

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