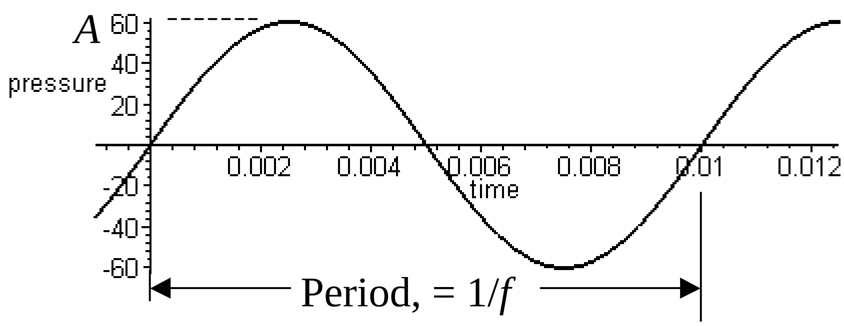
Mathematical Harmonies

Music is periodic variation in air pressure.

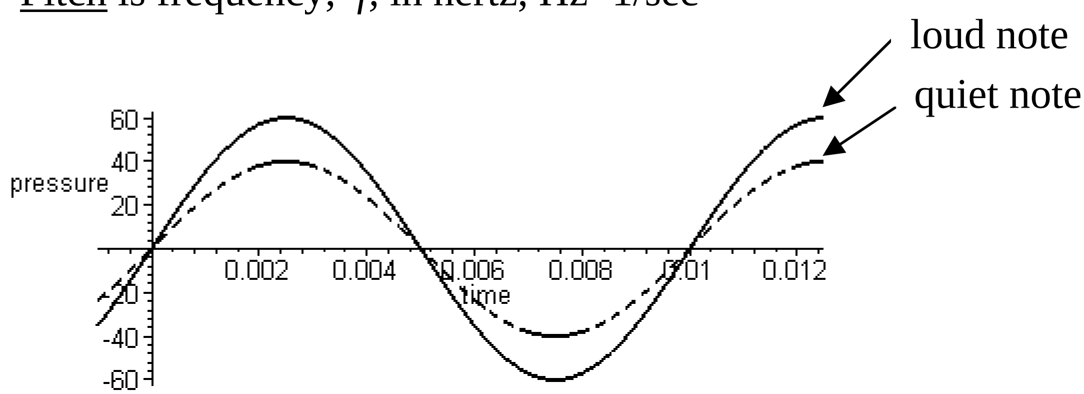
$$P = A \sin(2\pi f t)$$

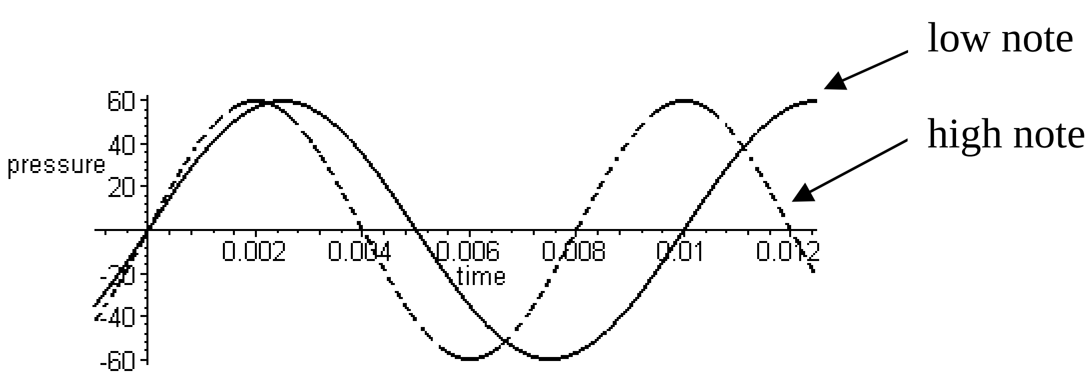
where: P pressure, in decibels or Pascals
t time, in seconds

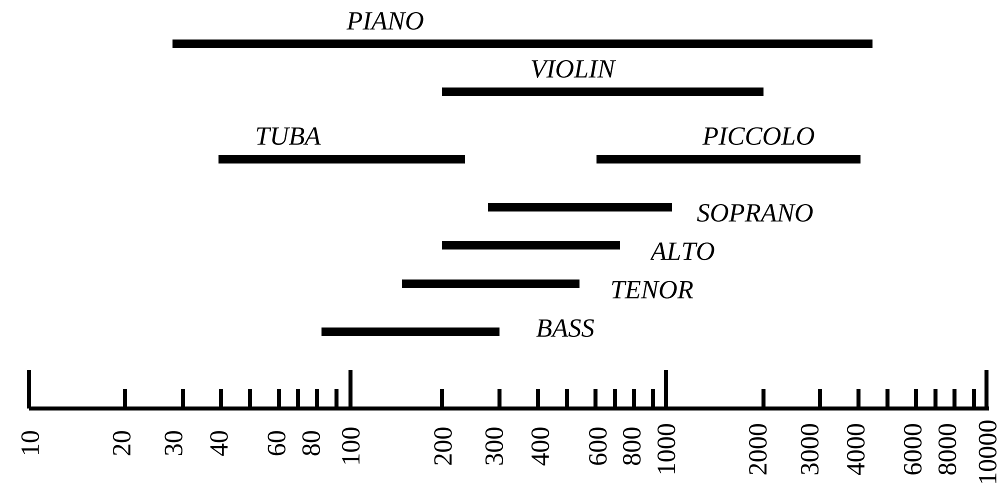


Sound has two characteristics:

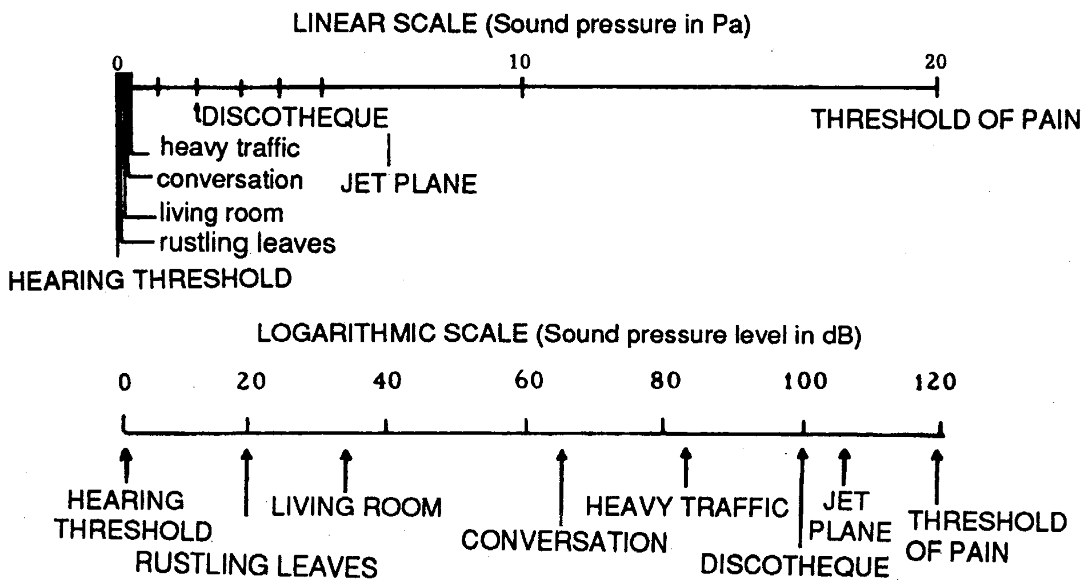
- <u>Volume</u> is amplitude, A, in Pascals or decibels
- <u>Pitch</u> is frequency, f, in hertz, Hz=1/sec







Frequency ranges of various instruments, in Hz. Audible frequencies range from 20 Hz to 20,000 Hz.



Linear scale: Pascals, Pa = N/m

Logarithmic scale: decibels, dB

$$p_{dB} = 20 * \log \frac{p_{Pa}}{2 \times 10^{-5}}$$

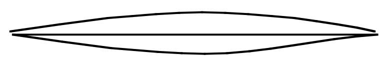
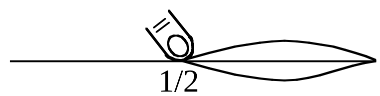
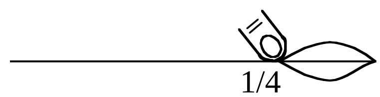
Frequency of a vibrating string:

$$frequency = \frac{1}{2 * length}\sqrt{\frac{tension}{thickness}}$$

We can change frequency in three ways:

1. Tighten the string:	↑ tension	↑ frequency
2. Use a thicker string:	↑ line density	↓ frequency
3. Use fingers on frets:	↓ length	↑ frequency

Specifically, halving the length will double the frequency.

Note	**Frequency**	**Diagram of vibrating string**
low low low A	f = 55 Hz	
low low A	f = 110 Hz	 1/2
low A	f = 220 Hz	 1/4
middle A	f = 440 Hz	 1/8

Octaves of a vibrating string.

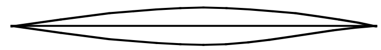
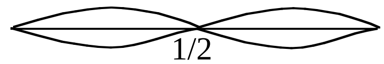
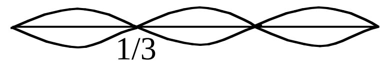
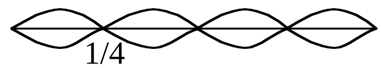
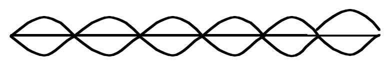
This sequence: 55, 110, 220, 440,Éis a geometric sequence.

A geometric sequence is a sequence where the previous term is multiplied by a constant. In this case, the constant is two.

Example: 2, 4, 8, 16, 32, É

The frequencies of octaves form a geometric sequence.

A string vibrates in many modes, called harmonics.

Note	Frequency	Harmonic	Diagram of string
low low low A	*f* = 55 Hz	fundamental	
low low A	*f* = 110 Hz	second	 1/2
low E	*f* = 165 Hz	third	 1/3
low A	*f* = 220 Hz	fourth	 1/4
middle $C^{\#}$	*f* = 275 Hz	fifth	 1/5
middle E	*f* = 330 Hz	sixth	 1/6
approx. middle G	*f* = 385 Hz	seventh	
middle A	*f* = 440 Hz	eighth	

The sequence: 55, 110, 165, 220, 275, És an arithmetic sequence.

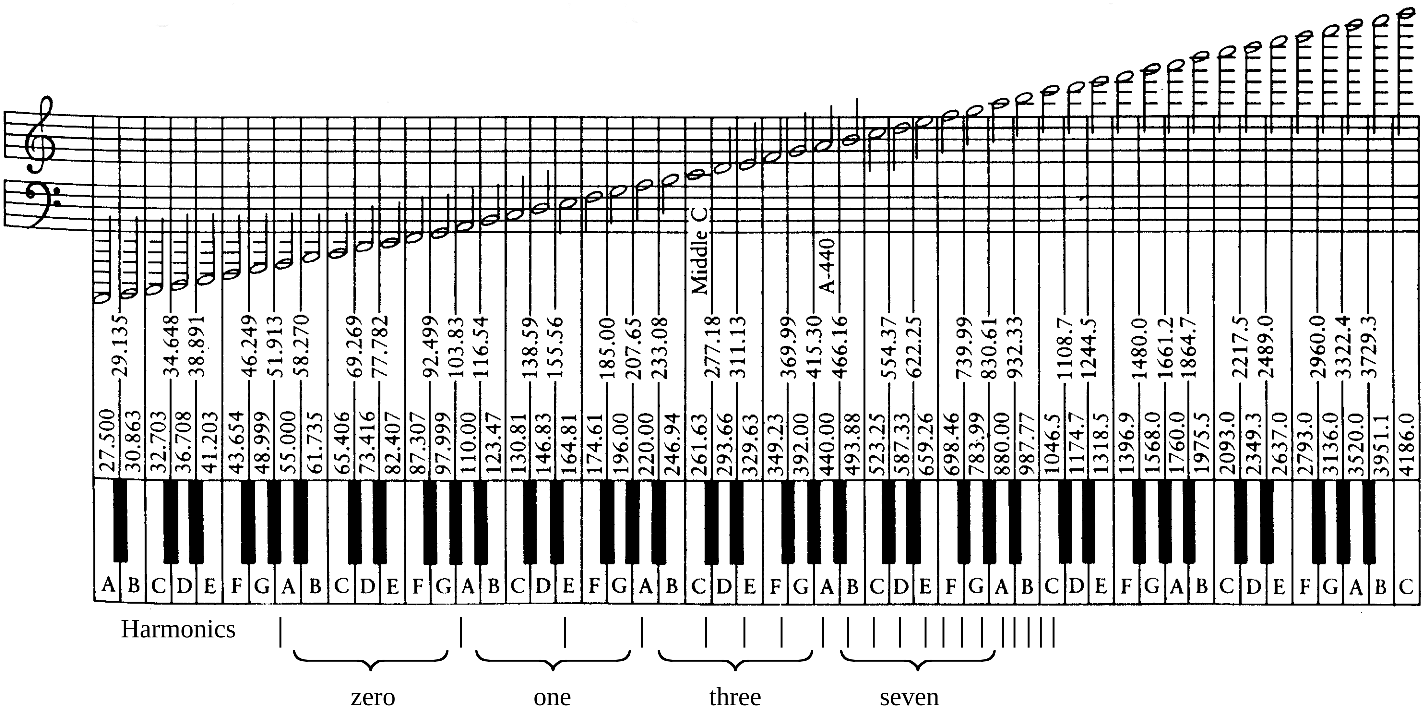
An arithmetic sequence is a sequence where a constant is added to the previous term. In this case, the constant is 55.

Example: 2, 4, 6, 8, 10, É

The frequencies of octaves form a geometric sequence.
The frequencies of harmonics form an arithmetic sequence.

Let us overlay an arithmetic sequence (harmonics) on a geometric sequence (the octaves):

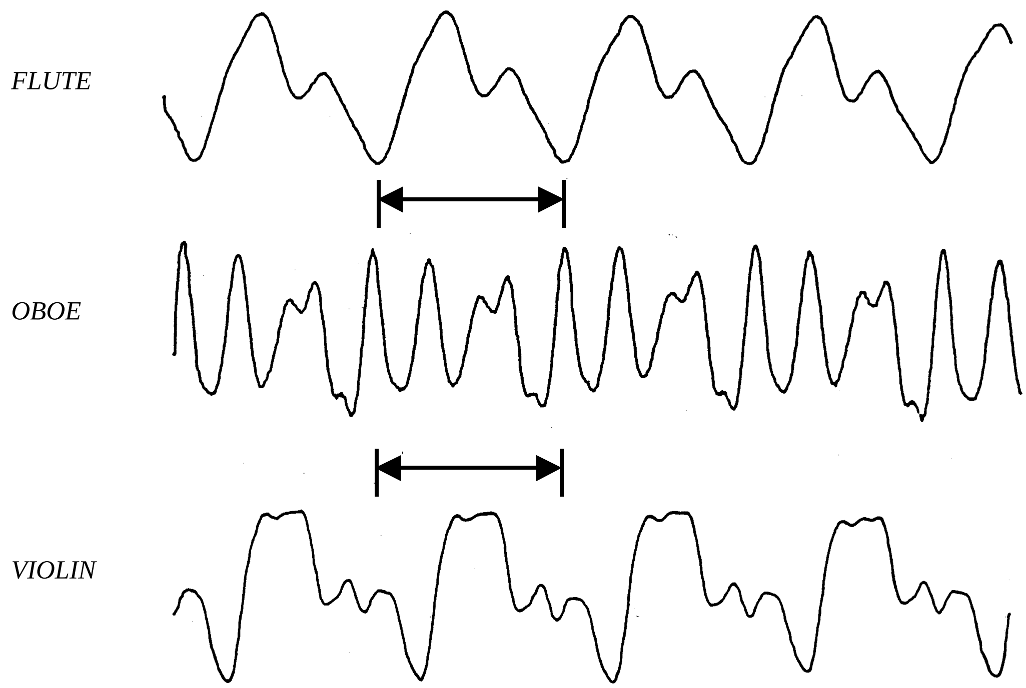
Arithmetic (harmonics)	2	4	6	8	10	12	14	16	18	20
Geometric (octaves)	2	4		8				16		
Number terms in between:	zero	one		three				seven		



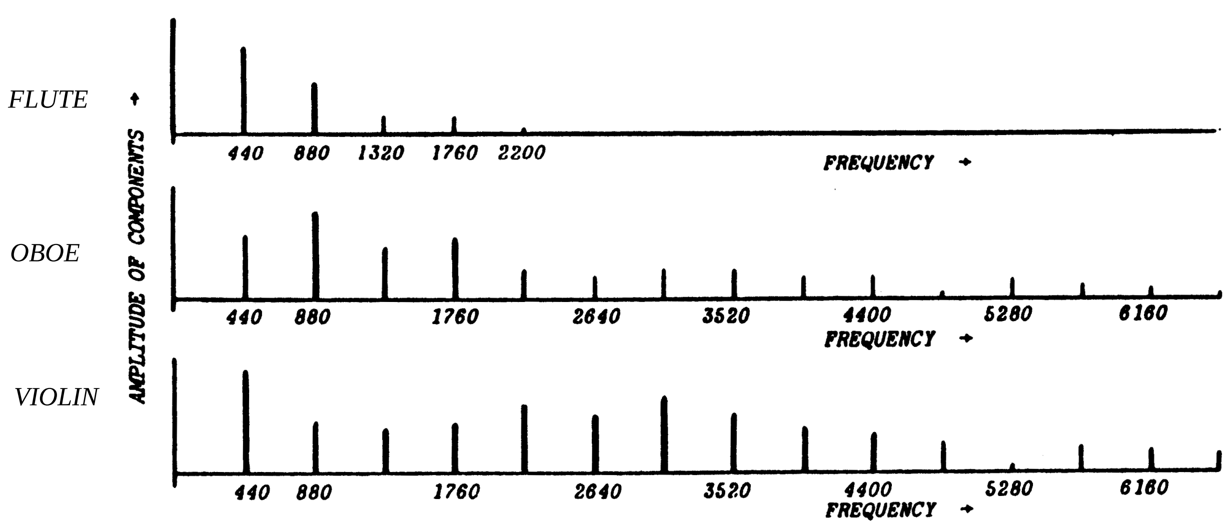
Harmonics of low low low A

Harmonics of Instruments

Pressure variations with time of a flute, oboe, and violin.

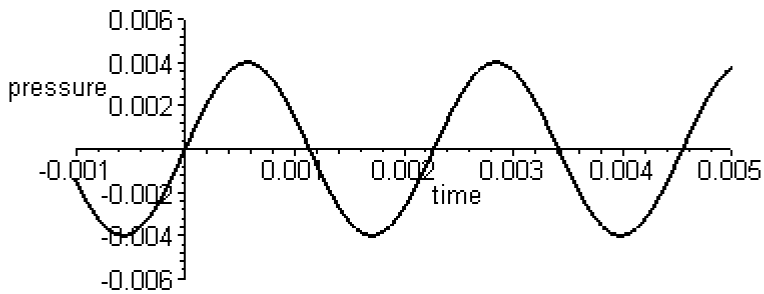


Amplitudes of the harmonics of a flute, oboe, and violin playing middle A.

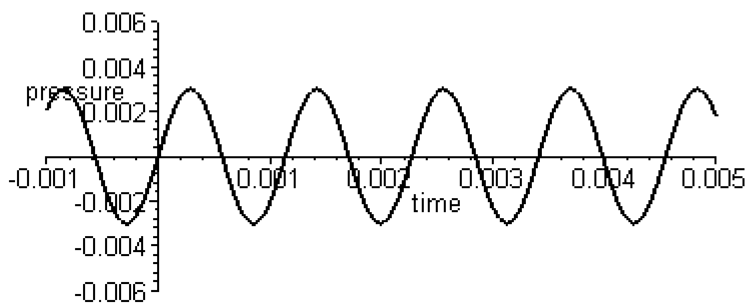


Build the pressure signature of a flute:

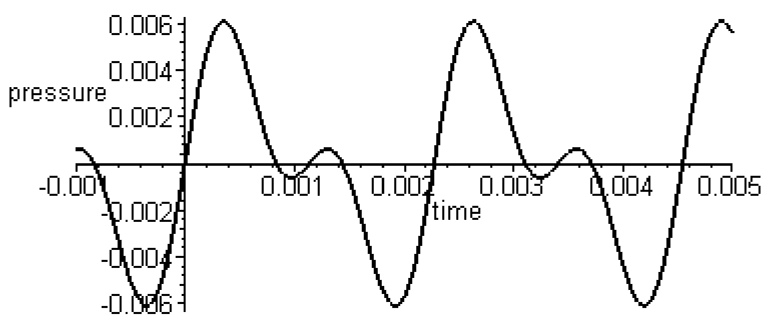
Fundamental: 440 Hz, 0.004 Pa = 46 dB



Second Harmonic: 880 Hz, 0.003 Pa = 43.5 dB

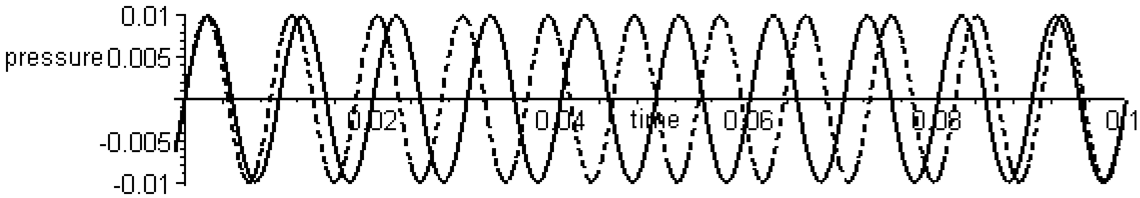


Sum of fundamental and second harmonic.

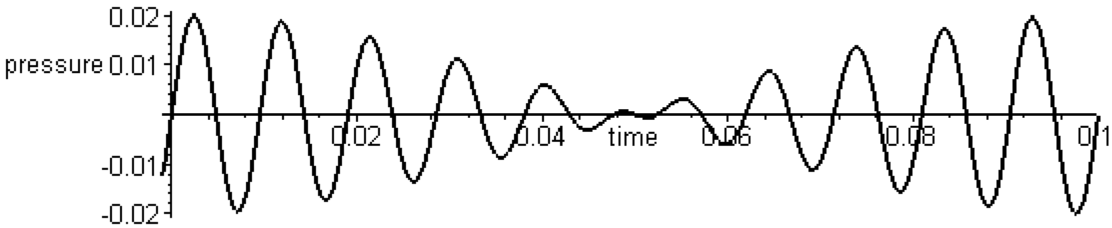


Superposition of two waves of slightly different frequency.

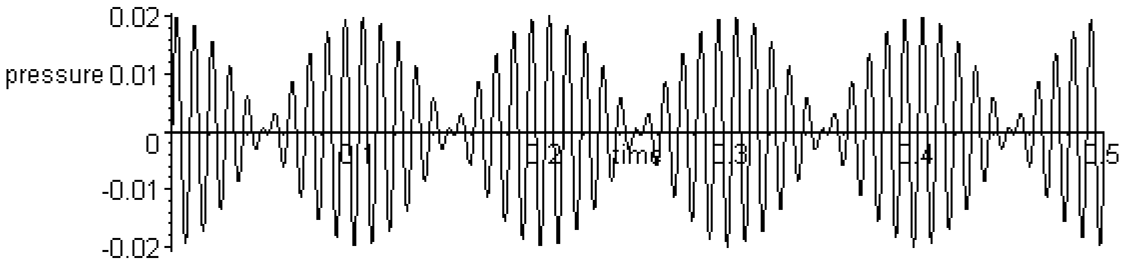
Two frequencies, 100 Hz and 110 Hz, both at 0.01 Pa



Summation of above frequencies.

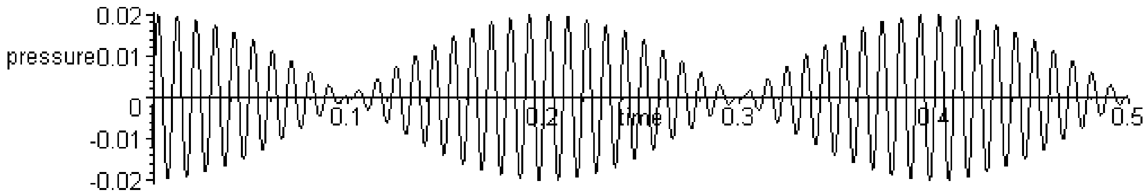


This pattern produces super-waves which are audible as beats.

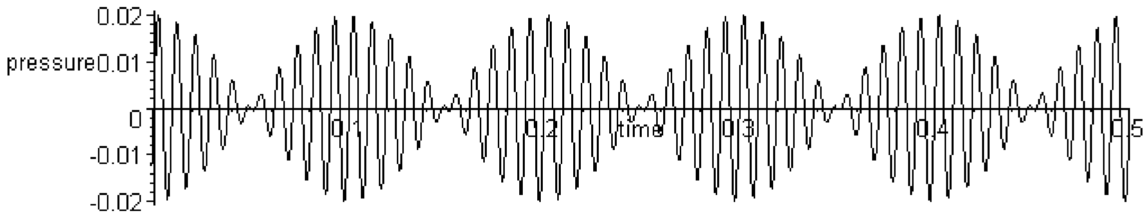


Beats from playing two notes with slightly different frequencies.

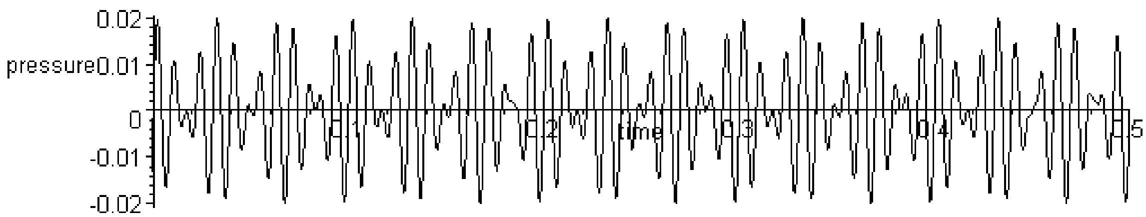
100 Hz and 105 Hz. Frequency separation: 0.05



100 Hz and 110 Hz. Frequency separation: 0.1

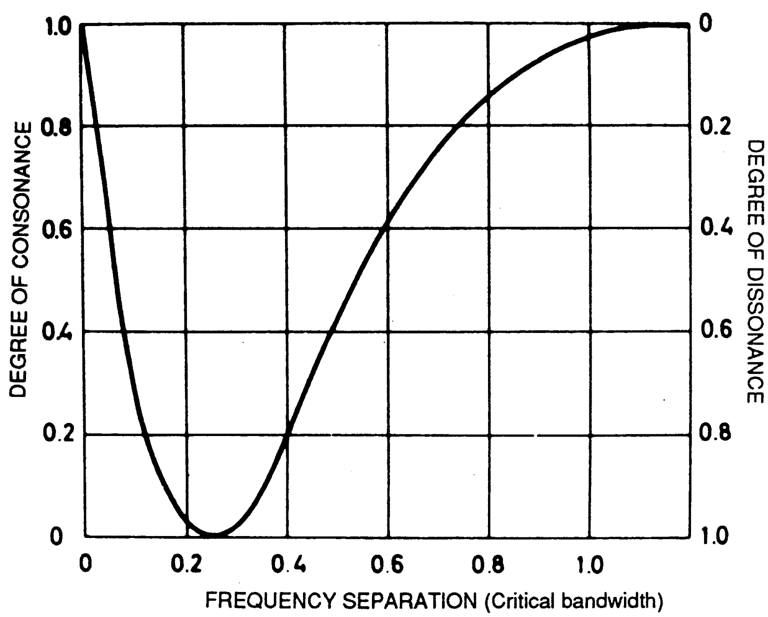


100 Hz and 130 Hz. Frequency separation: 0.3

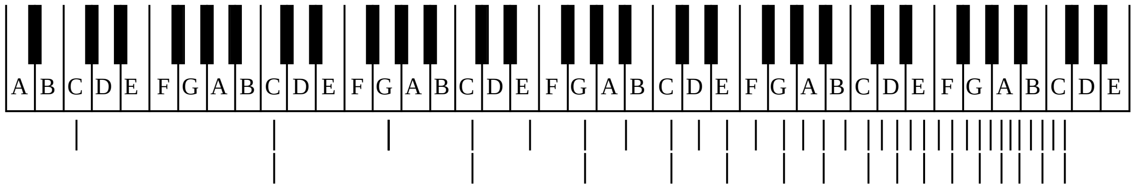


Conclusion:

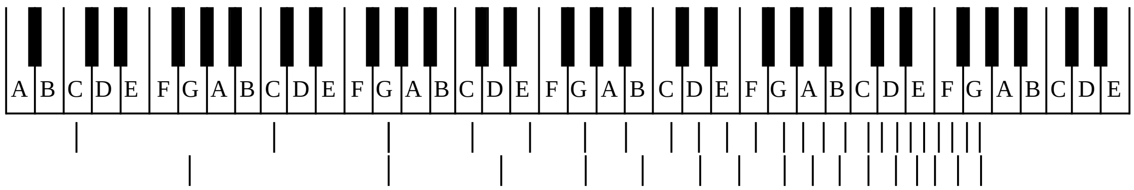
Frequencies close to each other create beats and sound bad (dissonance)



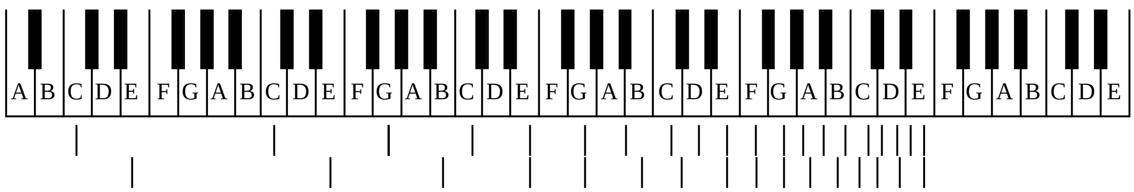
Octaves Harmonics of low low C and low C.



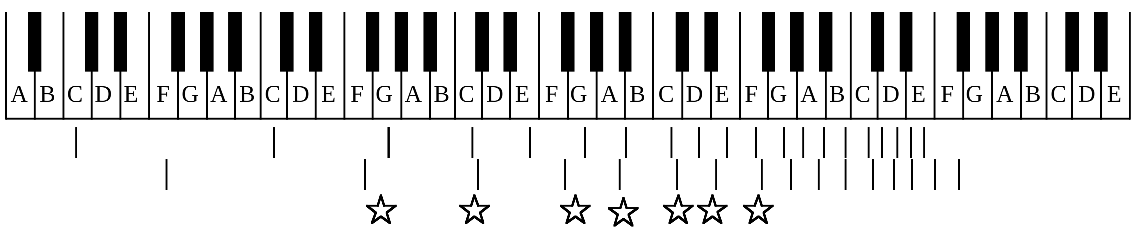
Fifth Harmonics of C and G.

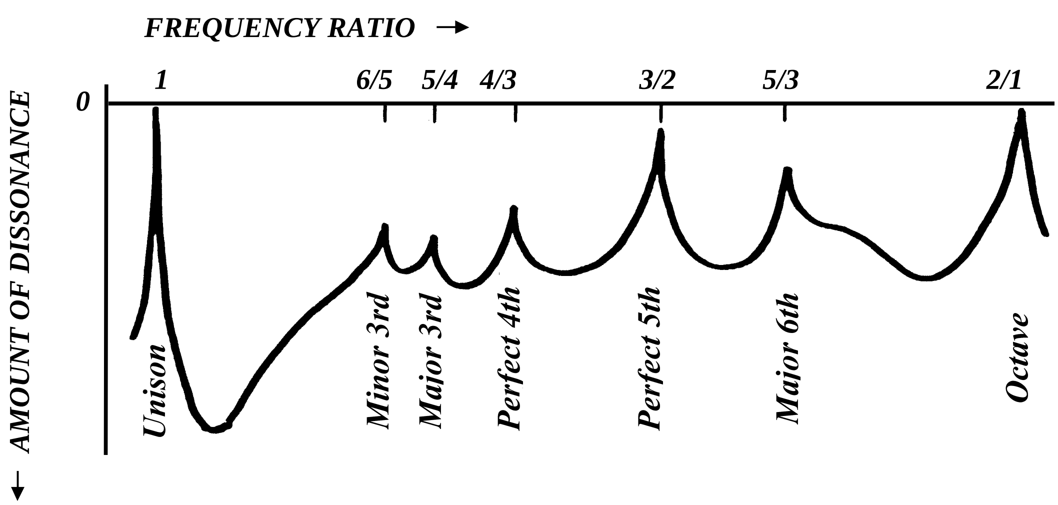


Third Harmonics of C and E.



Diminished Fifth Harmonics of C and F#.

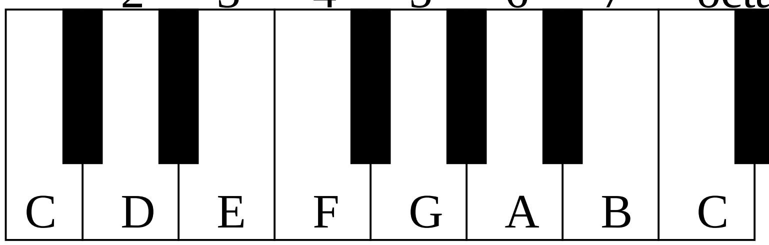




Just and Equal Temperament

Key of C

Interval from C:



Just Temperament

	C	D	E	F	G	A	B	C
Interval from C:		2nd	3rd	4th	5th	6th	7th	octave
Frequency ratio:	1	9/8	5/4	4/3	3/2	5/3	15/8	2
Frequency (Hz):	65.4	73.6	81.8	87.2	98.1	109	122.6	130.8

Ratio from one note to the next:

C→D	D→E	E→F	F→G	G→A	A→B	B→C
1.125	1.111	(1.067)	1.125	1.111	1.125	(1.067)

Equal Temperament

	C	D	E	F	G	A	B	C
Frequency (Hz):	65.4	73.4	82.4	87.3	98	110	123.5	130.8

Ratio from one note to the next:

C→D	D→E	E→F	F→G	G→A	A→B	B→C
1.122	1.122	(1.059)	1.122	1.122	1.122	(1.059)

Calculating Equal Temperament:

- There are 12 half steps in an octave, and an octave's frequency ratio is 2.

  So the frequency ratio of each half step is: $\sqrt[12]{2} = 1.059$

- There are 6 whole steps in an octave

  So the frequency ratio of each whole step is: $\sqrt[6]{2} = 1.122$